hep-th/0412103

Generalized Unitarity and One-Loop Amplitudes in $\mathcal{N} = 4$ Super-Yang-Mills

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arXiv:hep-th/0412103v4 31 Jan 2005

One-loop amplitudes of gluons in $\mathcal{N} = 4$ gauge theory can be written as linear combinations of known scalar box integrals with coefficients that are rational functions. In this paper we show how to use generalized unitarity to basically read off the coefficients. The generalized unitarity cuts we use are quadruple cuts. These can be directly applied to the computation of four-mass scalar integral coefficients, and we explicitly present results in next-to-next-to-MHV amplitudes. For scalar box functions with at least one massless external leg we show that by doing the computation in signature $(--++)$ the coefficients can also be obtained from quadruple cuts, which are not useful in Minkowski signature. As examples, we reproduce the coefficients of some one-, two-, and three-mass scalar box integrals of the seven-gluon next-to-MHV amplitude, and we compute several classes of three-mass and two-mass-hard coefficients of next-to-MHV amplitudes to all multiplicities.

December 2004

1. Introduction

One-loop amplitudes of gluons in supersymmetric field theories are four-dimensional cut constructible [1,2]. That is to say, they are completely determined by their unitarity cuts.

In the particular case of $\mathcal{N} = 4$ gauge theories, the amplitudes can be written as a sum over scalar box integrals with rational functions as coefficients [1,3]. Scalar box integrals are one-loop Feynman integrals in a scalar field theory with four external legs and four propagators. The scalar in the loop is massless, and the ones in the external legs could be massive. This gives four families classified by the number of massive legs. These are called one-, two-, three-, and four-mass scalar box integrals. Given that all scalar box integrals are known explicitly [3], the task of computing a given one-loop amplitude is reduced to that of finding the coefficients.

These coefficients were computed for all maximally helicity violating amplitudes in [1], for next-to-MHV six-gluon amplitudes in [2], for next-to-MHV seven-gluon amplitude with like helicity gluons adjacent in [4] and for all helicity configurations in [5].

One common feature of all these computations is that they are based on the use of unitarity cuts. The basic idea is to compute the discontinuity of the amplitude across a given branch cut in the kinematical space of invariants by adding all Feynman graphs with the same cut. The discontinuity is obtained by "cutting" two propagators. This sum of cut Feynman graphs reduces to the product of two tree-level amplitudes integrated over the Lorentz-invariant phase space of cut propagators. On the other hand, each of the scalar box integrals are also Feynman graphs whose discontinuity can also be computed by cutting two propagators. Therefore, combining the two different ways of computing the same discontinuity, one can get information about the coefficients.

One complication of all these approaches is that there are several scalar box integrals sharing a given branch cut. Therefore, one has to disentangle the information of several coefficients at once. This is done in [1,2,5] by using reduction techniques [6,7,8] and in [9,4] by using the holomorphic anomaly [10], which affects the action of differential operators that test localizations in twistor space [11].

In this paper we present a different way of computing the coefficients, using generalized unitarity. Even though several scalar box integrals can share the same branch cut, it turns out that their leading singularity is unique. For a detailed treatment of generalized unitarity and leading singularities of Feynman graphs, see Sections 2.9 and 2.2 of [12].

Therefore, by studying the discontinuity associated to the leading singularity, which we denote by Δ_{LS}, we can basically read off a given coefficient.

For a given Feynman graph, Δ_{LS} is obtained by cutting all propagators. In [13] the leading singularity of a three-mass triangle graph, whose discontinuity is computed by a triple cut, was used to compute its contribution to the $e^+e^- \to (\gamma^*, Z) \to \overline{q}qgg$ one-loop amplitude. In this paper, we use a quadruple cut to compute coefficients of four-mass box integrals in one-loop $\mathcal{N} = 4$ gluon amplitudes. We then turn our attention to scalar box integrals with at least one massless leg. In Minkowski space, Δ_{LS} for these box integrals, which is still a quadruple cut, does not give information about the coefficients, as we explain in section 2. One has to use at most triple cuts.[1] One disadvantage of triple cuts is that there might be several box integrals with the same singularity, and therefore several unknown coefficients will show up at once.

We find a way out by going to signature $(--++)$.[2] We show that in this signature all scalar box integral coefficients can be computed by a quadruple cut. This allows us to also read off the coefficients directly, since again only one coefficient contributes in a given cut. The quadruple cut integral is completely localized by the four delta functions of the cut propagators. It reduces to a product of four tree-level amplitudes, which can be easily computed by MHV diagrams [14,15,16]. This is a very simple way of computing any given coefficient in any one-loop $\mathcal{N} = 4$ gluon amplitude.

We illustrate this procedure by computing several coefficients of one-, two-, three- and four-mass box scalar integrals. For the first three cases we give examples for next-to-MHV amplitudes, including some results to all multiplicities, and for the last we give examples for an eight-gluon next-to-next-to-MHV (NNMHV) amplitude.

Motivated by the introduction of twistor string theory [11], there has recently been interest in the twistor-space localization of gauge theory amplitudes [14,17,10]. In particular, for supersymmetric gauge theories, the coefficients of box, triangle and bubble scalar integrals also exhibit simple twistor space structure [5,18]. At one-loop, this structure can be understood from the MHV diagram formulation [19].

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[1] Triple cuts were used in [5] to conclude that some coefficients in one-loop $\mathcal{N} = 4$ gluon amplitudes must vanish for special choices of helicity.

[2] Alternatively, it is perhaps more natural to drop the restriction of real momenta and work with the complexified Lorentz group. We thank E. Witten for pointing this out.

Since no four-mass coefficient has previously been presented in the literature, we consider its twistor space support. This is done by studying various differential operators acting on unitarity cuts.

This paper is organized as follows: In section 2, we discuss the generalized unitarity method in general and discuss how one can use quadruple cuts for all scalar box integrals in signature $(--++)$. In section 3, we present examples of each type of scalar box integral, starting with the four-mass, and then the three-, two-, and one-mass. In section 4, we give several examples of infinite classes of coefficients in next-to-MHV amplitudes. In the appendices, we consider in detail the discontinuity of the four-mass scalar box integral associated to branch cuts in all possible channels and use this to get consistency checks on the new coefficient obtained in section 3.

1.1. Preliminaries

One-loop $\mathcal{N} = 4$ amplitudes of gluons can be written as a linear combination of scalar box integrals with rational coefficients [3,1]. In this paper, we concentrate on the leading-color contribution, which is the part of the full amplitude proportional to $N\mathrm{Tr}\,(T^{a_1}\dots T^{a_n})$.

We write this schematically as follows:

$$A_{n:1} = \sum \left(\widehat{b} I^{1m} + \widehat{c} I^{2m\ e} + \widehat{d} I^{2m\ h} + \widehat{g} I^{3m} + \widehat{f} I^{4m} \right). \tag{1.1}$$

The integrals are defined in dimensional regularization as

$$I_4 = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2(\ell - K_1)^2(\ell - K_1 - K_2)^2(\ell + K_4)^2}. \tag{1.2}$$

The external momenta K_i are taken to be outgoing and are given by the sum of the momenta of consecutive external gluons, as shown in Figure 1. The labels $1m, 2m, 3m, 4m$ refer to the number of legs K_n such that $K_n^2 \neq 0$, or equivalently, the number of vertices in the box with more than one external gluon. For two-mass box integrals, there are two inequivalent arrangements of massive legs. Either they are adjacent ($I^{2m\ h}$) or they are diagonally opposite ($I^{2m\ e}$). All these integrals are UV finite but suffer from IR divergences, except for I^{4m} which is finite.

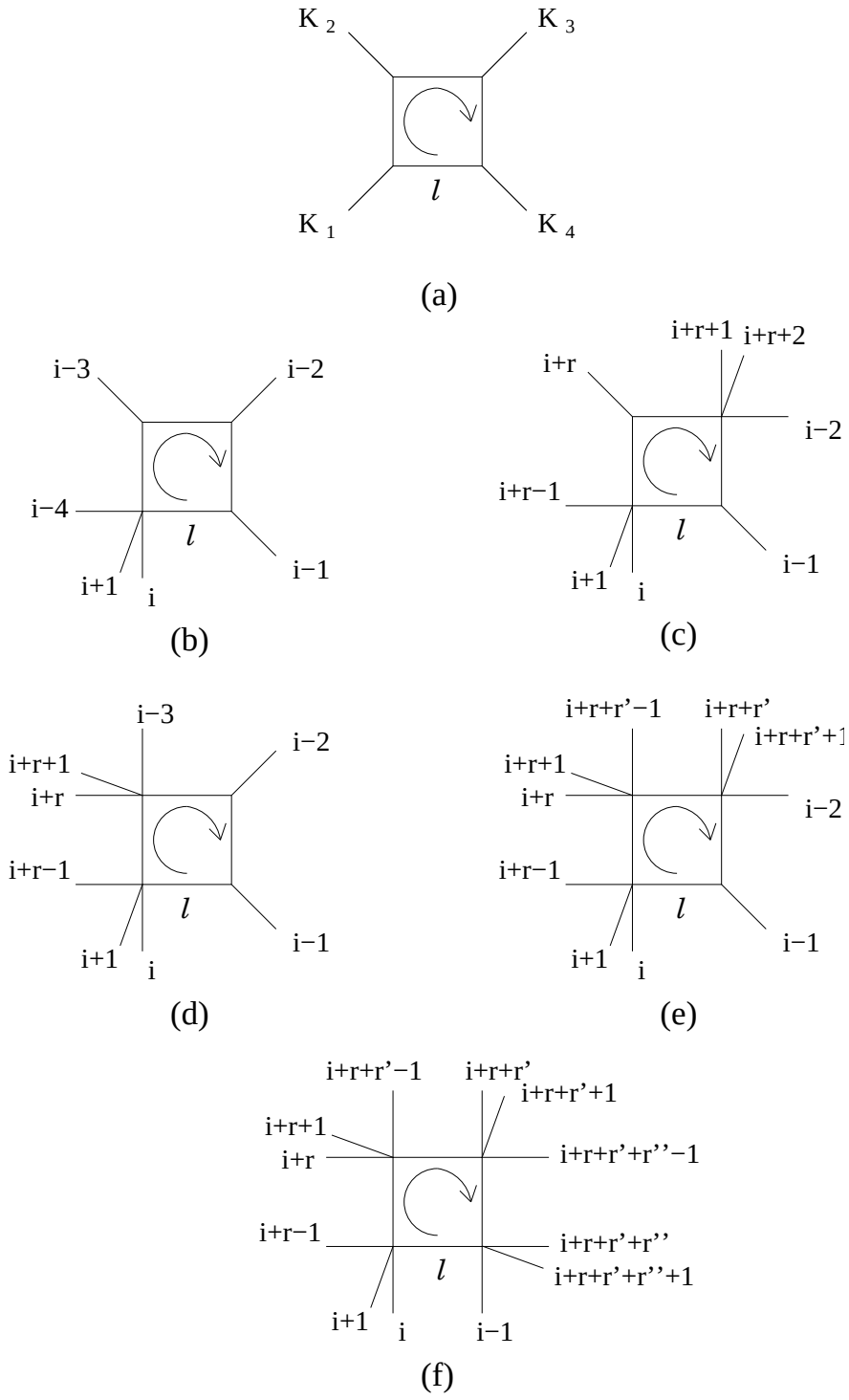

Fig. 1: Scalar box integrals. (a) The outgoing external momenta at each of the vertices are K_1, K_2, K_3, K_4, defined to correspond to sums of the momenta of gluons in the exact orientation shown. (b) One-mass $I^{1m}_{n:i}$. (c) Two-mass "easy" $I^{2m\ e}_{n:r;i}$. (d) Two-mass "hard" $I^{2m\ h}_{n:r;i}$. (e) Three-mass $I^{3m}_{n:r:r';i}$. (f) Four-mass $I^{4m}_{n:r:r':r";i}$.

From here on, we will drop the dimensional regularization parameter ϵ, because we will only deal with cuts that are finite. Moreover, we work in four-dimensional Minkowski space.

In the literature, it is common to write the amplitude in terms of scalar box functions. These functions are given in terms of the scalar box integrals as follows.

$$\begin{aligned} &I^{1m}_{4:i} = -2\frac{F^{1m}_{n:i}}{K^2_{41}K^2_{12}}, \quad I^{2m\ e}_{4:r;i} = -2\frac{F^{2m\ e}_{n:r;i}}{K^2_{41}K^2_{12} - K^2_1K^2_3}, \quad I^{2m\ h}_{4:r;i} = -2\frac{F^{2m\ h}_{n:r;i}}{K^2_{12}K^2_{41}}, \\ &I^{3m}_{4:r:r';i} = -2\frac{F^{3m}_{n:r:r';i}}{K^2_{41}K^2_{12} - K^2_1K^2_3}, \quad I^{4m}_{4:r:r':r";i} = -2\frac{F^{4m}_{n:r:r':r";i}}{K^2_{41}K^2_{12}\rho}. \end{aligned} \tag{1.3}$$

Here, and throughout the paper, we define

$$\begin{aligned} K_{mn} &= K_m + K_n, \\ \rho &= \sqrt{1 - 2\lambda_1 - 2\lambda_2 + (\lambda_1 - \lambda_2)^2}, \\ \lambda_1 &= \frac{K^2_1K^2_3}{K^2_{41}K^2_{12}}, \\ \lambda_2 &= \frac{K^2_2K^2_4}{K^2_{41}K^2_{12}}. \end{aligned} \tag{1.4}$$

Then we can alternatively write (1.1) as a linear combination of scalar box functions

$$A_{n:1} = \sum \left(bF^{1m} + cF^{2m\ e} + dF^{2m\ h} + gF^{3m} + fF^{4m} \right). \tag{1.5}$$

Each way of writing the amplitude has its own advantages and disadvantages. In (1.1), all coefficients are rational, but their twistor space support is not simple. In (1.5), the coefficient of the four-mass box function is not rational, for it contains a square root, but all coefficients have simple twistor space structure. For a discussion of the localization in twistor space of the four-mass scalar box function coefficient, see the appendices. For the rest of the body of the paper, we will work mainly with scalar box integrals and their coefficients as formulated in (1.1).

2. Generalized Unitarity and Quadruple Cuts

One-loop amplitudes in field theory have several singularities as complex functions of the kinematical invariants. In $\mathcal{N} = 4$ gauge theory, the singularities can only be those of the scalar box integrals (1.2) and of the coefficients in (1.1). Since the coefficients are

rational functions, they are not affected by branch cut singularities. Therefore one can get information about them by studying the discontinuities of the amplitude across the cuts.

In fact, most of the techniques for computing the coefficients efficiently are based on studying unitarity cuts [1,2,9,4,5]. The basic idea is to consider the branch cut singularity of the amplitude in a given channel. The discontinuity across this branch cut can be computed by a cut integral on both sides of the equation

$$A_{n:1} = \sum \left(\widehat{b} I^{1m} + \widehat{c} I^{2m\ e} + \widehat{d} I^{2m\ h} + \widehat{g} I^{3m} + \widehat{f} I^{4m} \right). \tag{2.1}$$

On the left hand side, one cuts two propagators of all Feynman integrals participating in this channel, while on the right hand side one cuts two propagators of scalar box functions.

By "cutting propagators" we mean the following. In Minkowski space, propagators in one loop integrals are defined by using Feynman's $i\epsilon$ prescription, i.e., $1/(P^2 + i\epsilon)$. This is equal to the principal value of $1/P^2$ plus $i\delta^{(+)}(P^2)$, where $(+)$ indicates a restriction to the future light-cone. Cutting a propagator means removing the principal part, i.e., replacing the propagator by $i\delta^{(+)}(P^2)$.

The sum over cut Feynman integrals on the left of (2.1) becomes an integral over a Lorentz invariant phase space of the product of two tree-level amplitudes. To be more explicit, consider the cut in the $(i, i+1, ..., j)$-channel,

$$C = \int d\mu \ A^{\rm tree}(\ell_1, i, ..., j, \ell_2) \ A^{\rm tree}(-\ell_2, j+1, ..., i-1, -\ell_1), \tag{2.2}$$

where $d\mu$ is the Lorentz invariant phase space measure for (ℓ_1, ℓ_2). It is given explicitly by

$$d\mu = \delta^{(+)}(\ell_1^2) \ \delta^{(+)}(\ell_2^2) \ \delta^{(4)}(\ell_1 + \ell_2 - P_{ij}), \tag{2.3}$$

where P_{ij} is the sum of the momenta of gluons from i to j.

If we denote the discontinuity across the branch cut of a given scalar box integral I by ΔI, then we have the following equation.

$$\begin{aligned} &\int d\mu A^{\rm tree}(\ell_1, i, ..., j, \ell_2) A^{\rm tree}(-\ell_2, j+1, ..., i-1, -\ell_1) = \\ &\sum \left(\widehat{b} \Delta I^{1m} + \widehat{c} \Delta I^{2m\ e} + \widehat{d} \Delta I^{2m\ h} + \widehat{g} \Delta I^{3m} + \widehat{f} \Delta I^{4m} \right). \end{aligned} \tag{2.4}$$

As discussed in the introduction, these equations have been used to get the coefficients for MHV amplitudes and six- and seven- gluon next-to-MHV amplitudes [1,2,9,4,5].

However, extracting the coefficients is not a simple task in general. The main reason is that several scalar box integrals share a given cut, and therefore their unknown coefficients enter in the equation at the same time.

Several approaches exist to extract the coefficients. One is based on reduction techniques that allow writing the integrand of (2.2) as a sum of terms that have the structure of cut scalar box functions [1,2]. Another method [9,4] uses operators that test localization in twistor space to get rational functions on both sides of the discontinuity of (2.1) and compare the pole structure.

It is the aim of this section to use higher order singularities to reduce the number of scalar box functions that enter in the generalization of (2.4). Ideally, we would like to find only one scalar box integral on the right hand side of (2.4).

This can easily be done for the four-mass scalar box integral. Consider the discontinuity associated to its leading singularity, Δ_{LS}. As mentioned in the introduction, this is computed from the integral

$$I^{4m} = \int d^4\ell \frac{1}{(\ell^2 + i\epsilon)((\ell - K_1)^2 + i\epsilon)((\ell - K_1 - K_2)^2 + i\epsilon)((\ell + K_4)^2 + i\epsilon)} \tag{2.5}$$

by cutting all four propagators:

$$\Delta_{LS} I^{4m} = \int d^4\ell \; \delta^{(+)}(\ell^2) \; \delta^{(+)}((\ell - K_1)^2) \; \delta^{(+)}((\ell - K_1 - K_2)^2) \; \delta^{(+)}((\ell + K_4)^2). \tag{2.6}$$

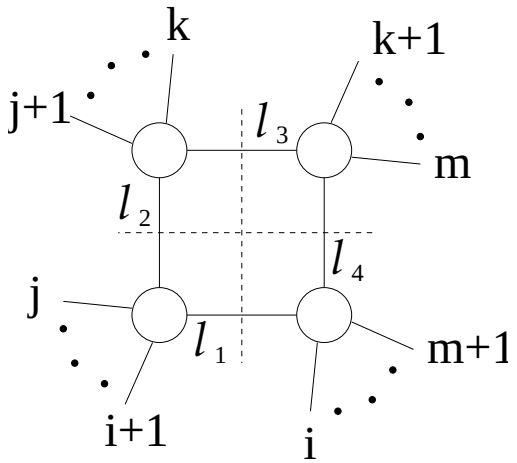

Fig. 2: A quadruple cut diagram. Momenta in the cut propagators flows clockwise and external momenta are taken outgoing. The tree-level amplitude A_1^{tree}, for example, has external momenta $i+1, ..., j, \ell_2, \ell_1$.

Now it turns out that no other box integral in (2.1) shares the same singularity. Therefore, the generalization of (2.4) is

$$\int d^4\ell\ \delta^{(+)}(\ell^2)\ \delta^{(+)}((\ell-K_1)^2)\ \delta^{(+)}((\ell-K_1-K_2)^2)\ \delta^{(+)}((\ell+K_4)^2)\times \\ A_1^{\rm tree}A_2^{\rm tree}A_3^{\rm tree}A_4^{\rm tree} = \widehat{f}\Delta_{LS}I^{4m}, \tag{2.7}$$

where $A_n^{\rm tree}$ is the tree-level amplitude at the corner with total external momentum K_n. See fig. 2.

Note that since there are four delta functions, and ℓ is a vector in four dimensions, then the integral is localized and equals a jacobian $\mathcal{J} = (4K_{41}^2K_{12}^2\rho)^{-1}$. Moreover, the same jacobian appears on both sides of (2.7), and we find that

$$\widehat{f} = \frac{1}{|\mathcal{S}|}\sum_{\mathcal{S},J} n_J(A_1^{\rm tree}A_2^{\rm tree}A_3^{\rm tree}A_4^{\rm tree}), \tag{2.8}$$

where the sum is over the possible spins J of internal particles and the solution set $\mathcal{S}$ of ℓ's for the localization equations, $|\mathcal{S}|$ is the number of these solutions, and n_J is the number of particles of spin J.

This equation gives the coefficient $\widehat{f}$ of any four-mass box function in any one-loop amplitude, as we discuss in more detail in section 2.1.

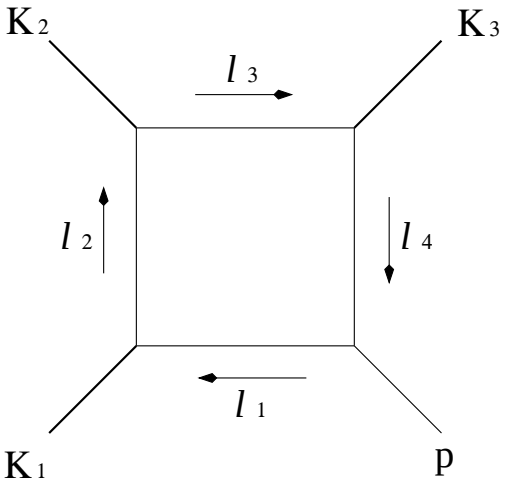

Fig. 3: Scalar box integral with at least one massless leg. We use thick lines for massive legs and thin lines for massless legs.

The natural question at this point is whether the same thing can be done for other scalar box integral coefficients. It turns out that in Minkowski space this is not possible. Recall that all other scalar box integrals contain at least one massless vertex. This means that at least one of the four tree-level amplitudes in the quadruple cut must be a three-gluon amplitude. One problem is that all three-gluon amplitudes vanish on-shell, as will become clear in the discussion below.

For definiteness, consider the three-mass scalar box integral in Figure 3. Let us denote the momenta of the three-gluon vertex by ℓ_1, ℓ_4 and p. We have to impose that all three vectors be lightlike and that $\ell_4 = \ell_1 + p$. From this it is easy to see that necessary conditions for all three vectors to be lightlike are $\ell_1 \cdot p = \ell_4 \cdot p = \ell_1 \cdot \ell_4 = 0$. In Minkowski signature one can go to a frame where $p = (E, 0, 0, E)$. Then it is easy to see from $\ell_1 \cdot p = 0$ that $\ell_1 = \alpha p$. Likewise, $\ell_4 = \beta p$. This means that all three gluons are collinear. Momentum conservation then implies that $\beta = \alpha + 1$.

On the other hand, from $\ell_2 = \ell_1 - K_1$ and $\ell_2^2 = 0$ we can compute α, and from $\ell_3 = \ell_4 + K_3$ we get β. They are given by

$$\alpha = \frac{K_1^2}{2K_1 \cdot p}, \qquad \beta = -\frac{K_3^2}{2K_3 \cdot p}. \tag{2.9}$$

Now we can ask if the equation $\beta = \alpha + 1$ is satisfied. It is not difficult to see that this equation is equivalent to $K_1^2 K_3^2 - (K_1 + K_2)^2 (K_1 + p)^2 = 0$. Note that the left hand side is precisely the denominator in (1.3). This means that this singularity is a pole, as opposed to the usual branch cut singularity. The discontinuity associated to it is a delta function,[3] reflecting the fact that only three of the four delta functions suffice to perform the integral.

This implies that the quadruple cut equation (2.7) is in this case given by

$$\int d\mu \; A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}} = \widehat{g} \; \delta(K_1^2 K_3^2 - (K_1 + K_2)^2 (K_1 + p)^2). \tag{2.10}$$

Note that the left hand side is trivially zero, for A_4 is a three-gluon tree-level amplitude on-shell which vanishes in Minkowski space. Then we find that

$$\widehat{g} \; \delta(K_1^2 K_3^2 - (K_1 + K_2)^2 (K_1 + p)^2) = 0, \tag{2.11}$$

[3] In one real variable, the statement that the discontinuity associated to a pole is a delta function is expressed in the familiar relation $\frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon} = 2\pi i \delta(x)$.

and therefore the coefficient must have a zero at the support of the delta function. Indeed, recall that the relation between the box integral coefficient $\widehat{g}$ and the box function coefficient g is given by $\widehat{g} = \frac{1}{2}(K_1^2 K_3^2 - (K_1 + K_2)^2 (K_1 + p)^2) g$.

This proves that the information about g disappears from the equation, and the quadruple cut cannot be used to learn anything about $\widehat{g}$, except for the presence of the factor $\frac{1}{2}(K_1^2 K_3^2 - (K_1 + K_2)^2 (K_1 + p)^2)$.

If we insist on using Minkowski signature, we have to study triple cuts.[4] The disadvantage is that in most cases, more than one box integral will share the same singularity. The situation is better than with the usual cut in (2.4), but it is still complicated in most cases.

2.1. Quadruple Cut in Signature $(--++)$

The main source of the problem for scalar box integrals with at least one massless leg is the presence of the three-gluon vertex. The fact that this always vanishes in Minkowski space made impossible the extraction of their coefficients from the quadruple cut.

As explained carefully in section 3 of [11], three-gluon amplitudes with helicities $(++-)$ and $(--+)$ do not necessarily vanish in other signatures. In particular, motivated by the study of a string theory with target twistor space [11], Witten considered the three-gluon amplitude in signature $(--++)$. It turns out that this is precisely what is needed in order to use the quadruple cuts to read off all coefficients.

We now review the analysis done in [11]. In four dimensions we can write a null vector as a bispinor, $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$. The inner product of two null vectors p and q is given by $2p \cdot q = \langle \lambda_p \ \lambda_q \rangle [\widetilde{\lambda}_p \ \widetilde{\lambda}_q]$, where the brackets represent the natural inner products of spinors of positive and negative chirality.

A tree-level three-gluon amplitude with helicity $(++-)$ or $(--+)$ is given respectively by [20]

$$A_3^{\text{tree}}(p^+, q^+, r^-) = \frac{[\widetilde{\lambda}_p \ \widetilde{\lambda}_q]^3}{[\widetilde{\lambda}_r \ \widetilde{\lambda}_p][\widetilde{\lambda}_q \ \widetilde{\lambda}_r]}, \qquad A_3^{\text{tree}}(p^-, q^-, r^+) = \frac{\langle \lambda_p \ \lambda_q \rangle^3}{\langle \lambda_r \ \lambda_p \rangle \langle \lambda_q \ \lambda_r \rangle}. \tag{2.12}$$

In Minkowski space and for real momenta, λ_p and $\widetilde{\lambda}_p$ are complex but not independent, $\widetilde{\lambda}_p = \pm \overline{\lambda}_p$. Therefore, when $p \cdot q = 0$, this means that both $\langle p \ q \rangle$ and $[p \ q]$ vanish. This implies that both amplitudes in (2.12) also vanish.

[4] To be precise, all unitarity cut analyses require complexified momenta. We thank L. Dixon for this clarification.

On the other hand, in signature $(--++)$ and for real momenta, both λ_p and $\widetilde{\lambda}_p$ are real and independent. Therefore, $2p \cdot q = \langle \lambda_p \, \lambda_q \rangle [\widetilde{\lambda}_p \, \widetilde{\lambda}_q] = 0$ has two solutions. Either $\langle \lambda_p \, \lambda_q \rangle = 0$ or $[\widetilde{\lambda}_p \, \widetilde{\lambda}_q] = 0$.

Since momentum conservation implies that $p \cdot q = p \cdot r = q \cdot r = 0$, we find that if $\langle \lambda_p \, \lambda_q \rangle = 0$, then $\langle \lambda_p \, \lambda_r \rangle = 0$ and $\langle \lambda_r \, \lambda_q \rangle = 0$ must also hold. Therefore all three λ's are proportional. Likewise if we choose $[\widetilde{\lambda}_p \, \widetilde{\lambda}_q] = 0$, then all three $\widetilde{\lambda}$'s are proportional.

Now it is clear that if we are faced with a $(++-)$ tree amplitude, we should choose all λ's to be proportional, and then the corresponding amplitude in (2.12) will not vanish. Likewise, if we are faced with a $(--+)$, we should choose all $\widetilde{\lambda}$'s to be proportional.

Having solved the problem of the vanishing of the amplitudes, we have to deal with the meaning of a "unitarity cut" in signature $(--++)$. We certainly cannot offer a formal theory here, but some interesting developments have appeared in the literature about field theories in $(--++)$ signature. For a recent review, see [21].

Here we take a more operational approach in order to compute the coefficients. Since the coefficients are written in terms of invariant products of spinors, once they are computed they can be used in Minkowski space.

Consider the quadruple cut measure in Minkowski space. Recall that each delta function is actually given by

$$\delta^{(+)}(P^2) = \vartheta(E_P)\delta(P^2), \tag{2.13}$$

where E_P is the zeroth component of P^μ. Both sides of (2.7) contain the same measure and in particular the same factors of $\vartheta(x)$. Since the integral is localized, one can drop the $\vartheta(x)$ factors on both sides. We then take as our definition of a quadruple cut in signature $(--++)$ the following:

$$\begin{aligned} &\int d^4\ell \; \delta(\ell^2) \; \delta((\ell - K_1)^2) \; \delta((\ell - K_1 - K_2)^2) \; \delta((\ell + K_4)^2) A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}} = \\ &\widehat{g} \int d^4\ell \; \delta(\ell^2) \; \delta((\ell - K_1)^2) \; \delta((\ell - K_1 - K_2)^2) \; \delta((\ell + K_4)^2). \end{aligned} \tag{2.14}$$

Here we use $\widehat{g}$ to represent a generic coefficient in (2.1) with at least one massless leg.

In (2.14) the delta functions are ordinary Dirac delta functions. One can solve for $\ell_{a\dot{a}}$, taking into account that solutions for which one or more of the tree-level amplitudes vanish do not contribute. Therefore we find that the coefficient must be equal to

$$\widehat{g} = \frac{1}{|\mathcal{S}|} \sum_{\mathcal{S},J} n_J A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}, \tag{2.15}$$

where again the sum is over all solutions.

In the rest of the paper, we illustrate this procedure in detail.

3. Examples

The previous section demonstrated that coefficients of box integrals may be computed from the formula

$$\widehat{a}_\alpha = \frac{1}{|\mathcal{S}|} \sum_{\mathcal{S},J} n_J A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree}, \tag{3.1}$$

where J is the spin of a particle in the $\mathcal{N}=4$ multiplet and $\mathcal{S}$ is the set of all solutions of the on-shell conditions for the internal lines,

$$\mathcal{S} = \{\ \ell \mid \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0\}. \tag{3.2}$$

In this section, we present a variety of applications of the formula (3.1).

The explicit covariant solution of (3.2) for the vector ℓ is given by:

$$\begin{aligned}
\ell &= \beta_1 K_1 + \beta_2 K_2 + \beta_3 K_4 + \beta_4 P; \\
P_\mu &= \epsilon_{\mu\nu\rho\sigma} K_1^\nu K_2^\rho K_4^\sigma, \\
\beta_1 &= \frac{1}{2P^2}\left[(K_2\cdot K_4)(-(K_1\cdot K_4)(2(K_1\cdot K_2)+K_2^2) + K_1^2(K_2\cdot K_4))\right. \\
&\quad \left. +(2(K_1\cdot K_2)^2 - (K_1^2 + (K_1\cdot K_4))K_2^2 + (K_1\cdot K_2)(K_2^2 + (K_2\cdot K_4)))K_4^2\right], \\
\beta_2 &= \frac{1}{2P^2}\left[(K_1\cdot K_4)(2(K_1\cdot K_2)(K_1\cdot K_4) + (K_1\cdot K_4)K_2^2 - K_1^2(K_2\cdot K_4))\right. \\
&\quad \left. -(-(K_1\cdot K_2)(K_1\cdot K_4) + K_1^2((K_1\cdot K_2) + K_2^2 + (K_2\cdot K_4)))K_4^2\right], \\
\beta_3 &= \frac{1}{2P^2}\left[(K_1\cdot K_2)(-(K_1\cdot K_4)K_2^2 + K_1^2(K_2\cdot K_4))\right. \\
&\quad \left. -(K_1\cdot K_2)^2(2(K_1\cdot K_4)+K_4^2) + K_1^2K_2^2((K_1\cdot K_4)+(K_2\cdot K_4)+K_4^2)\right], \\
\beta_4 &= \frac{\pm K_{12}^2 K_{41}^2 \rho}{4P^2}.
\end{aligned} \tag{3.3}$$

In particular, $|\mathcal{S}| = 2$, and the two solutions S_+, S_- are related by a change of sign of ρ, which appears in β_4. Since there is a sum over the two solutions S_+ and S_-, the result for $\widehat{a}_\alpha$ is seen to be rational, as it must be [3].[5]

We begin with the four-mass box integral, where the solution (3.3) can be used directly in the formula (3.1).[6]

[5] For one-, two-, and three-mass coefficients, ρ turns out to be rational, so each solution is individually rational.

[6] The formula (3.3) is implicitly written in signature $(--++)$. To perform a calculation with external momenta in Minkowski space, one must Wick-rotate $\epsilon_{\mu\nu\rho\sigma} \to i\epsilon_{\mu\nu\rho\sigma}$.

For all other box integrals, we must work in the signature $(--++)$, as described in section 2. As we have seen, the helicity configuration determines how to solve for the spinor components of the cut propagators.

It is worth noting that for all these cases, there are still just the two solutions (3.3); in fact, a given solution determines whether the holomorphic or antiholomorphic spinors are proportional at each three-gluon vertex. This relation is illustrated in Figure 4, for each family of box functions with a three-gluon vertex.

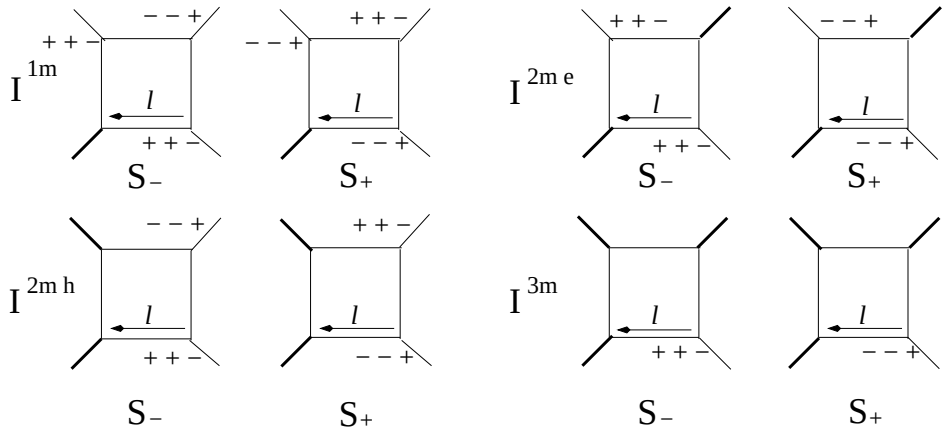

Fig. 4: Possible helicity assignments at three-gluon vertices, as derived from the two solutions S_+, S_- given in (3.2). The assignment $++-$ means that the solution dictates that the holomorphic spinors λ are all proportional at this vertex; the assignment $--+$ means that the solution dictates that the antiholomorphic spinors $\widetilde{\lambda}$ are proportional. The three helicities can then be distributed freely on the three legs at that vertex.

In practice, however, it is useful to solve for spinor components directly, rather than using (3.3) as written. This is because the four tree-level amplitudes in (3.1) can be easily computed from Parke-Taylor formulas [20] or MHV diagrams [14,15], which involve spinor products.

In the three-mass example, we demonstrate the judicious solution of spinor components. In the two-mass-easy example, we show that overall momentum conservation is an important constraint on the solutions of spinor components. In the two-mass-hard example, we show that any pair of adjacent three-gluon corners must come with opposite helicity configurations (one each of $++-$ and $--+$), which is consistent with Figure 4. This restriction figures into our one-mass example also, where we choose to compute a box integral whose quadruple cut has a next-to-MHV amplitude at one corner.

The four-mass coefficient presented here is new, but our examples in this section for three-,two-, and one-mass coefficients are all for a next-to-MHV seven-gluon amplitude that was computed in [4] and [5]. Our formulas reproduce the ones in those papers exactly.

In section 4, we will present several examples of infinite classes of coefficients for n-gluon next-to-MHV amplitudes.

3.1. Four-Mass Box Integral Coefficients

Let us illustrate our solution by computing the coefficients $\widehat{f}_1, \widehat{f}_2$ of the scalar box functions $I^{4m}_{8:2:2:2;1}$ and $I^{4m}_{8:2:2:2;2}$ for the helicity configuration $(1^-, 2^-, 3^-, 4^-, 5^+, 6^+, 7^+, 8^+)$. We can see immediately that $\widehat{f}_1 = 0$, because any possible assignment of helicities for the internal lines, one of the four tree-level amplitudes in (3.1) always vanishes.

The box diagram for $I^{4m}_{8:2:2:2;2}$ is shown in Figure 5.

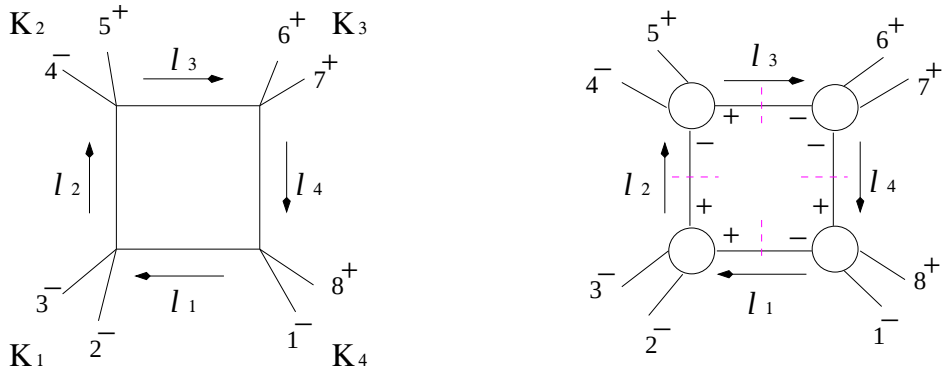

Fig. 5: The box function $I^{4m}_{8:2:2:2;2}$ for the helicity configuration $(1^-, 2^-, 3^-, 4^-, 5^+, 6^+, 7^+, 8^+)$ with internal momenta $\ell_1, \ell_2, \ell_3, \ell_4$, and its associated quadruple cut.

This helicity configuration allows only gluons to circulate in the loop. Moreover, each vertex is MHV, so it is straightforward to write the coefficient (3.1) in terms of Parke-Taylor formulas [20].

$$\begin{aligned}
\widehat{f}_2 &= \frac{1}{2} A^{\rm tree}((-\ell_4)^+, 8^+, 1^-, \ell_1^-)\times \\
&\qquad \times A^{\rm tree}((-\ell_1)^+, 2^-, 3^-, (-\ell_2)^+) A^{\rm tree}(\ell_2^-, 4^-, 5^+, (-\ell_3)^+) A^{\rm tree}(\ell_3^-, 6^+, 7^+, \ell_4^-) \\
&= \frac{1}{2} \sum_{S_+, S_-} \frac{\langle 1\ \ell_1\rangle^3}{\langle \ell_1\ \ell_4\rangle\langle \ell_4\ 8\rangle\langle 8\ 1\rangle} \frac{\langle 2\ 3\rangle^3}{\langle 3\ \ell_2\rangle\langle \ell_2\ \ell_1\rangle\langle \ell_1\ 2\rangle} \frac{\langle \ell_2\ 4\rangle^3}{\langle 4\ 5\rangle\langle 5\ \ell_3\rangle\langle \ell_3\ \ell_2\rangle} \frac{\langle \ell_4\ \ell_3\rangle^3}{\langle \ell_3\ 6\rangle\langle 6\ 7\rangle\langle 7\ \ell_4\rangle} \\
&= \frac{1}{2} \sum_{S_+, S_-} \frac{[6\ 7]^3 \langle 1|\ell_1\ \ell_2|4\rangle^3}{\langle 8\ 1\rangle[2\ 3]\langle 4\ 5\rangle\langle 5|\ell_3\ \ell_4\ \ell_1|2][3|\ell_2\ \ell_3|6]\langle 8|\ell_4|7]}.
\end{aligned} \tag{3.4}$$

Further information about four-mass box integrals, consistency checks for this coefficient, and twistor space structure may be found in the appendices.

3.2. Three-Mass Example

Consider the three-mass box scalar integral for the seven-gluon next-to-MHV amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$ shown in Figure 6.

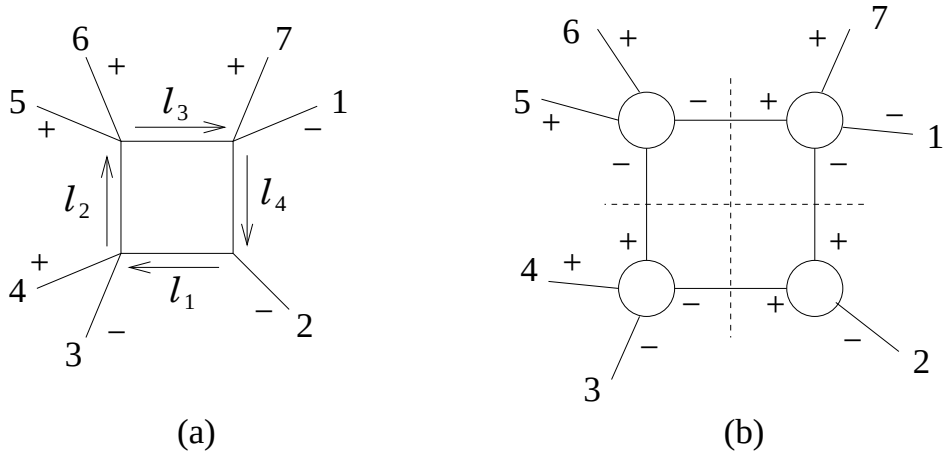

Fig. 6: A three-mass scalar box integral for the next-to-MHV amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$.

As explained in section 2, for the three-gluon amplitude with helicities $(+ + -)$,we should choose the λ's to be proportional so that the amplitude does not vanish. We make this choice in solving the equations $p_2 \cdot \ell_1 = p_2 \cdot \ell_4 = \ell_1 \cdot \ell_4 = 0$, which can be written as $\langle \ell_1\ 2\rangle[\ell_1\ 2] = \langle \ell_4\ 2\rangle[\ell_4\ 2] = \langle \ell_1\ \ell_4\rangle[\ell_1\ \ell_4] = 0$.

We choose to write each four gluon amplitude in the generalized unitarity cut diagram as a mostly minus MHV amplitude. This gives

$$\widehat{g}_3 = \frac{1}{2}\left(\frac{[\ell_1\ \ell_4]^3}{[\ell_1\ 2][2\ \ell_4]}\right)\left(\frac{[4\ \ell_2]^3}{[\ell_2\ \ell_1][\ell_1\ 3][3\ 4]}\right)\left(\frac{[5\ 6]^3}{[6\ \ell_3][\ell_3\ \ell_2][\ell_2\ 5]}\right)\left(\frac{[\ell_3\ 7]^3}{[7\ 1][1\ \ell_4][\ell_4\ \ell_3]}\right). \tag{3.5}$$

Now we explain how to solve λ_{ℓ_1} and $\widetilde{\lambda}_{\ell_1}$ from light-cone conditions. First, using $0 = 2\ell_1 \cdot p_2 = \langle 2\ \ell_1\rangle[2\ \ell_1]$, we get $\langle 2\ \ell_1\rangle = 0$ or $[2\ \ell_1] = 0$. To get a nonzero contribution, we choose to solve $\langle 2\ \ell_1\rangle = 0$, so that $\lambda_{\ell_1} = \alpha\lambda_2$. Now we can solve for α:

$$\begin{aligned} 2\ell_1 \cdot (p_3 + p_4) = (p_3 + p_4)^2 &= -\alpha\langle 2|(3+4)|\ell_1]; \\ \alpha &= \frac{-(p_3 + p_4)^2}{\langle 2|(3+4)|\ell_1]}. \end{aligned} \tag{3.6}$$

To solve for $\widetilde{\lambda}_{\ell_1}$, we use the equation $\ell_3^2 = (\ell_1 - (p_3 + p_4 + p_5 + p_6))^2 = 0$, i.e.,

$$\begin{aligned} 2\ell_1 \cdot (p_3 + p_4 + p_5 + p_6) = (p_3 + p_4 + p_5 + p_6)^2 = -\alpha\langle 2|(3+4+5+6)|\ell_1] \\ (p_3 + p_4 + p_5 + p_6)^2\langle 2|(3+4)|\ell_1] = (p_3 + p_4)^2\langle 2|(3+4+5+6)|\ell_1] \end{aligned} \tag{3.7}$$

Note that this is not enough to fix the whole $\widetilde{\lambda}_{\ell_1}$, but it fixes its direction. This is good enough to compute (3.5), since the coefficient has degree zero in $\widetilde{\lambda}_{\ell_1}$.

Then we do the same for ℓ_4 and compute λ_{ℓ_4} and $\widetilde{\lambda}_{\ell_4}$ up to a scale.

After this is done, we choose to write

$$[\ell_2\ \bullet] = \frac{\langle 2|\ell_2|\bullet]}{\langle 2\ \ell_2\rangle}, \qquad [\ell_3\ \bullet] = \frac{\langle 2|\ell_3|\bullet]}{\langle 2\ \ell_3\rangle}. \tag{3.8}$$

where $\bullet$ represents any external gluon, ℓ_1 or ℓ_4. It is easy to see that all denominators will drop out from the fact that the coefficient has degree zero in ℓ_2 and ℓ_3.

This gives for the coefficient (3.5) the following:

$$\widehat{g}_3 = \frac{1}{2}\left(\frac{[\ell_1\ \ell_4]^3}{[\ell_1\ 2][2\ \ell_4]}\right)\left(\frac{\langle 2|\ell_2|4]^3}{\langle 2|\ell_2|\ell_1][\ell_1\ 3][3\ 4]}\right)\left(\frac{[5\ 6]^3}{\langle 2|\ell_3|6]\langle 2|\ell_3\cdot\ell_2|2\rangle\langle 2|\ell_2|5]}\right)\left(\frac{\langle 2|\ell_3|7]^3}{[7\ 1][1\ \ell_4]\langle 2|\ell_3|\ell_4]}\right), \tag{3.9}$$

where we will substitute for ℓ_2 and ℓ_3 using

$$\ell_2 = \ell_1 - p_3 - p_4, \qquad \ell_3 = \ell_4 + p_1 + p_7. \tag{3.10}$$

The coefficient of the associated three-mass scalar box function was computed by two different methods in [4,5], and it is given by

$$\begin{aligned} g_3 &= -2\frac{\widehat{g}_3}{(p_3+p_4+p_5+p_6)^2(p_2+p_3+p_4)^2 - (p_3+p_4)^2(p_7+p_1)^2} \\ &= -\frac{\langle 1\ 2\rangle^3\langle 2\ 3\rangle^3[5\ 6]^3}{\langle 7\ 1\rangle\langle 3\ 4\rangle\langle 2|3+4|5]\langle 2|7+1|6](\langle 7\ 1\rangle\langle 2|3+4|1] - t_2^{[3]}\langle 7\ 2\rangle)(t_3^{[4]}\langle 2\ 4\rangle - \langle 3\ 4\rangle\langle 2|7+1|3])}. \end{aligned} \tag{3.11}$$

It is easy to check that the two formulas (3.9),(3.11) give exactly the same answer for $\widehat{g}_3$.

3.3. Two-Mass-Easy Example

Consider the coefficient c_4 in the amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$. The box integral is shown in Figure 7.

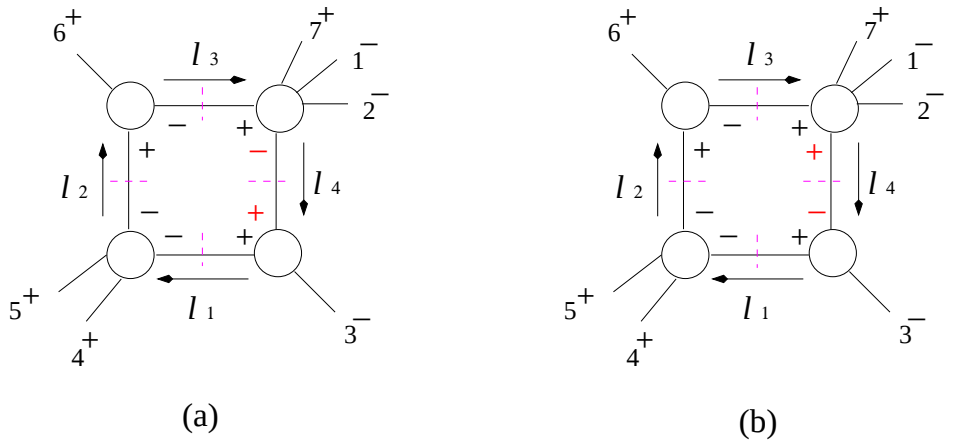

Fig. 7: Quadruple cut of a two-mass-easy scalar box integral for the amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$.

Notice that there are two possibly nonvanishing helicity assignments for the cut propagators. Here it is possible to solve for the spinor components of $(\ell_1, \ell_2, \ell_3, \ell_4)$, up to a scale for each, for both helicity configurations. However, the solution for configuration (b) found from this procedure fails to satisfy momentum conservation at the "massive" corners. Therefore, there is only one true solution, namely configuration (a). From it we find

$$\widehat{c}_4 = \frac{1}{2}\left(\frac{\langle \ell_2\ \ell_1\rangle^3}{\langle \ell_1\ 4\rangle\langle 4\ 5\rangle\langle 5\ \ell_2\rangle}\right)\left(\frac{[\ell_2\ 6]^3}{[6\ \ell_3][\ell_3\ \ell_2]}\right)\left(\frac{[\ell_3\ 7]^3}{[7\ 1][1\ 2][2\ \ell_4][\ell_4\ \ell_3]}\right)\left(\frac{[\ell_1\ \ell_4]^3}{[\ell_4\ 3][3\ \ell_1]}\right). \tag{3.12}$$

Then

$$c_4 = -2\frac{\widehat{c}_4}{(p_4+p_5+p_6)^2(p_3+p_4+p_5)^2 - (p_4+p_5)^2(p_7+p_1+p_2)^2}, \tag{3.13}$$

which agrees with the results given in [4,5].

3.4. Two-Mass-Hard Example

Consider the coefficient $d_{3,2}$ in the amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$. The box integral is shown in Figure 8.

There are two possibly nonvanishing helicity assignments for the cut propagators. For configuration (b), it is not possible to solve for the spinor components of $(\ell_1, \ell_2, \ell_3, \ell_4)$. The reason is that from the p_7 corner, we derive the condition that $\lambda_{\ell_3}, \lambda_{\ell_4}$ and λ_7 are all proportional, and similarly from the p_1 corner, we derive that $\lambda_{\ell_4}, \lambda_{\ell_1}$ and λ_1 are all proportional. But this means that λ_7 is proportional to λ_1, which is not true of generic

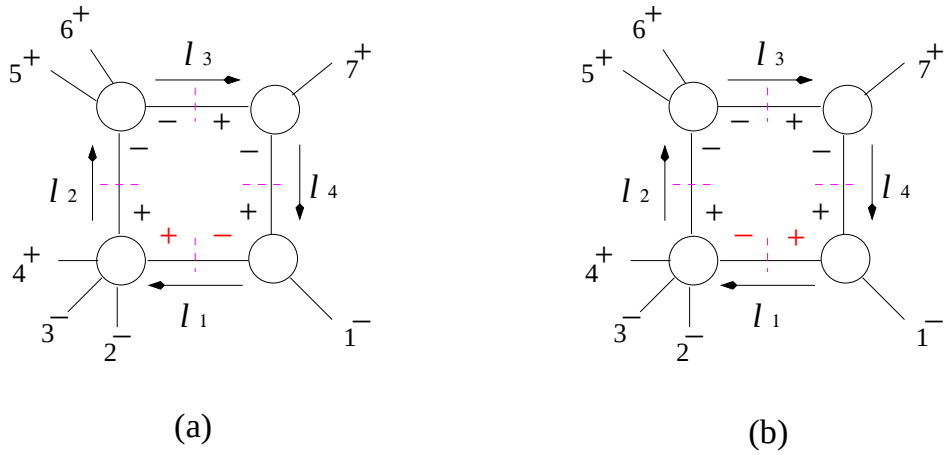

Fig. 8: Quadruple cut of a two-mass-hard scalar box integral for the amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$.

external momenta. The lesson here is that adjacent three-gluon box corners must have opposite helicity types in order to give a nonvanishing product of amplitudes.

For the helicity configuration (a), there is no such obstacle to solving for the spinor components of $(\ell_1, \ell_2, \ell_3, \ell_4)$ up to a scale, because the conditions are that $\lambda_{\ell_3}, \lambda_{\ell_4}$ and λ_7 are all proportional, but it is the antiholomorphic spinors $\widetilde{\lambda}_{\ell_4}, \widetilde{\lambda}_{\ell_1}$ and $\widetilde{\lambda}_1$ that are proportional for the other corner. Once the solution is obtained, the coefficient $\widehat{d}_{3,2}$ is given by

$$\widehat{d}_{3,2} = \frac{1}{2}\left(\frac{\langle \ell_1\ 1\rangle^3}{\langle 1\ \ell_4\rangle\langle \ell_4\ \ell_1\rangle}\right)\left(\frac{[\ell_3\ 7]^3}{[\ell_3\ \ell_4][\ell_4\ \ell_7]}\right)\left(\frac{\langle \ell_3\ \ell_2\rangle^3}{\langle \ell_2\ 5\rangle\langle 5\ 6\rangle\langle 6\ \ell_3\rangle}\right)\left(\frac{\langle 2\ 3\rangle^3}{\langle 3\ 4\rangle\langle 4\ \ell_2\rangle\langle \ell_2\ \ell_1\rangle\langle \ell_1\ 2\rangle}\right). \tag{3.14}$$

Then

$$d_{3,2} = -2\frac{\widehat{d}_{3,2}}{(p_5 + p_6 + p_7)^2 (p_7 + p_1)^2}, \tag{3.15}$$

which agrees with the results of [4,5].

3.5. One-Mass Example

Even for the one-mass box functions, where the internal momenta might seem to be overconstrained, we can solve for the coefficients using generalized unitarity. Consider the coefficient b_2 in the amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$. The box integral is shown in Figure 9. We have applied the lesson learned in the previous subsection, that adjacent three-gluon box corners must have opposite helicity types in order to give a nonvanishing product of amplitudes. Thus only the two helicity configurations shown in the diagram might contribute to the coefficient.

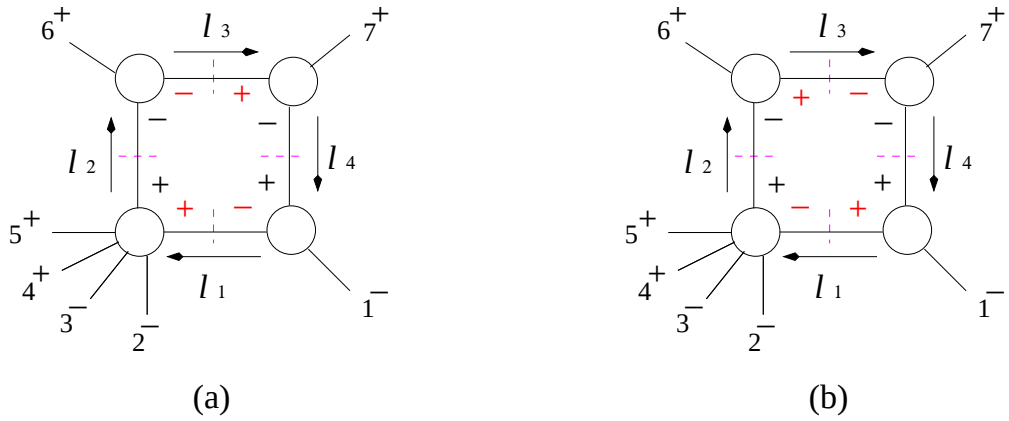

Fig. 9: Quadruple cut of a one-mass scalar box integral for the amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$.

For the configuration (a), we solve for the spinor components of $(\ell_1, \ell_2, \ell_3, \ell_4)$ while requiring that

$$[\ell_2\ 6] = [\ell_3\ 6] = [\ell_2\ \ell_3] = 0, \quad \langle \ell_3\ 7\rangle = \langle \ell_4\ 7\rangle = \langle \ell_3\ \ell_4\rangle, \quad [\ell_4\ 1] = [\ell_1\ 1] = [\ell_4\ \ell_1] = 0. \tag{3.16}$$

Then we substitute this solution into the product of tree amplitudes:

$$\widehat{b}_2^{(a)} = \frac{1}{2}\left(\frac{\langle 2\ 3\rangle^3}{\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ \ell_2\rangle\langle \ell_2\ \ell_1\rangle\langle \ell_1\ 2\rangle}\right)\left(\frac{\langle \ell_2\ \ell_3\rangle^3}{\langle \ell_2\ 6\rangle\langle 6\ \ell_3\rangle}\right)\left(\frac{[\ell_3\ 7]^3}{[7\ \ell_4][\ell_4\ \ell_3]}\right)\left(\frac{\langle 1\ \ell_1\rangle^3}{\langle \ell_1\ \ell_4\rangle\langle \ell_4\ 1\rangle}\right). \tag{3.17}$$

For the configuration (b), we solve for the spinor components of $(\ell_1, \ell_2, \ell_3, \ell_4)$ while requiring that

$$\langle \ell_2\ 6\rangle = \langle \ell_3\ 6\rangle = \langle \ell_2\ \ell_3\rangle = 0, \quad [\ell_3\ 7] = [\ell_4\ 7] = [\ell_3\ \ell_4], \quad \langle \ell_4\ 1\rangle = \langle \ell_1\ 1\rangle = \langle \ell_4\ \ell_1\rangle = 0. \tag{3.18}$$

We use this solution in the product of tree amplitudes, which now includes an NMHV amplitude [22,23]:

$$\begin{aligned} A_6^{\text{tree}}((-\ell_1)^-, 2^-, 3^-, 4^+, 5^+, \ell_2^+) = \Bigg[&\frac{\beta^2}{t_{5\ell_2(-\ell_1)}s_{5\ell_2}s_{\ell_2(-\ell_1)}s_{23}s_{34}} \\ &+ \frac{\gamma^2}{t_{\ell_2(-\ell_1)2}s_{\ell_2(-\ell_1)}s_{(-\ell_1)2}s_{34}s_{45}} + \frac{\beta\gamma t_{45\ell_2}}{s_{45}s_{5\ell_2}s_{\ell_2(-\ell_1)}s_{(-\ell_1)2}s_{23}s_{34}}\Bigg], \\ \beta &= [5\ \ell_2]\langle 2\ 3\rangle\langle \ell_1|5+\ell_2|4], \\ \gamma &= [4\ 5]\langle \ell_1\ 2\rangle\langle 3|4+5|\ell_2], \\ s_{ij} &= \langle i\ j\rangle[i\ j], \\ t_{ijk} &= \langle i\ j\rangle[i\ j] + \langle i\ k\rangle[i\ k] + \langle j\ k\rangle[j\ k]. \end{aligned} \tag{3.19}$$

The contribution of this configuration to the coefficient is

$$\widehat{b}_2^{(b)} = \frac{1}{2} A_6^{\rm tree}((-\ell_1)^-, 2^-, 3^-, 4^+, 5^+, \ell_2^+) \times \left(\frac{[6\ \ell_3]^3}{[\ell_3\ \ell_2][\ell_2\ 6]}\right) \left(\frac{\langle \ell_4\ \ell_3\rangle^3}{\langle \ell_3\ 7\rangle\langle 7\ \ell_4\rangle}\right) \left(\frac{[\ell_1\ \ell_4]^3}{[\ell_4\ 1][1\ \ell_1]}\right). \tag{3.20}$$

Then

$$b_2 = -2 \frac{\widehat{b}_2^{(a)} + \widehat{b}_2^{(b)}}{(p_6 + p_7)^2 (p_7 + p_1)^2,} \tag{3.21}$$

which agrees with the results of [4,5]. We note that in [4], this coefficient b_2 was given in terms of a seven-gluon tree amplitude and eight other coefficients (using a relation derived from the infrared singular behavior of a certain unitarity cut). Here, the quadruple cut allows us to obtain an explicit formula for b_2 directly.

One can easily obtain b_2 for the n-gluon next-to-MHV amplitude $(1^-, 2^-, 3^-, 4^+, ..., n^+)$ by substituting the tree amplitude $A_{n-1}^{\rm tree}((-\ell_1)^-, 2^-, 3^-, 4^+, ..., (n-2)^+, \ell_2^+)$ for $A_6^{\rm tree}$ in (3.20). Compact expressions for that amplitude have been given in [14,16].

4. All-Multiplicity Examples for NMHV Amplitudes

In this section we present some examples of using quadruple cuts to compute some classes of coefficients for n-gluon next-to-MHV amplitudes. Here we substitute the actual solutions for the cut propagators, so that our final formulas are given in terms of the external momenta only.

4.1. All-Multiplicity Examples of Three-Mass Coefficients

In this subsection we present three classes of three-mass coefficients for next-to-MHV n-gluon amplitudes. The first two are depicted in Figure 10.

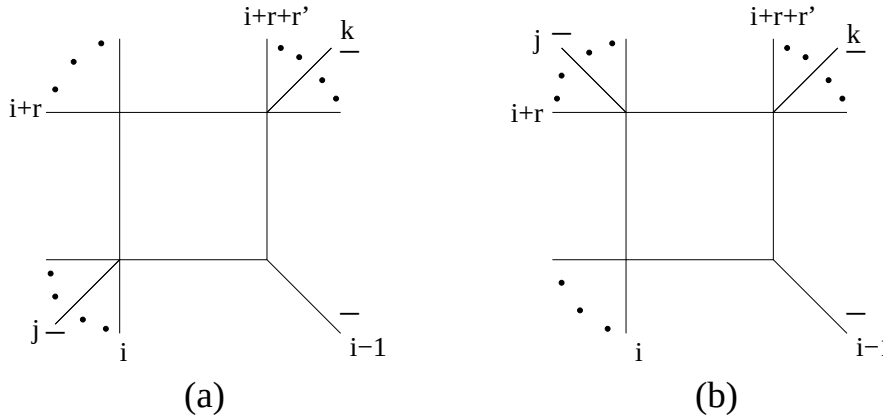

Fig. 10: Two families of three-mass scalar box integrals in an NMHV configuration.

Consider first the three-mass scalar box integral coefficients $\widehat{g}_{n:r:r';i}$, where the three negative helicities are $j, k, i-1$, and j is in the vertex of $(i, ..., i+r-1)$ and k is in the vertex of $(i+r+r', ..., i-2)$. See Figure 10(a). Using (3.1), we can write down

$$\widehat{g}_{n:r:r';i} = \frac{\langle i-2\ i-1\rangle\langle i-1\ i\rangle\langle i+r-1\ i+r\rangle\langle i+r+r'-1\ i+r+r'\rangle}{2\prod_{s=1}^{n}\langle s\ s+1\rangle}\frac{[\ell_1\ \ell_4]^3}{[\ell_4\ i-1][i-1\ \ell_1]}$$
$$\times \frac{\langle \ell_1\ j\rangle^4}{\langle \ell_2\ \ell_1\rangle\langle \ell_1\ i\rangle\langle i+r-1\ \ell_2\rangle}\frac{\langle \ell_3\ \ell_2\rangle^3}{\langle \ell_2\ i+r\rangle\langle i+r+r'-1\ \ell_3\rangle}\frac{\langle k\ \ell_4\rangle^4}{\langle \ell_3\ i+r+r'\rangle\langle i-2\ \ell_4\rangle\langle \ell_4\ \ell_3\rangle}. \tag{4.1}$$

We simplify the above formula by noticing that

$$\begin{aligned}\langle p|K_2\cdot K_1|i-1\rangle &= \langle p\ \ell_3\rangle[\ell_3\ \ell_2]\langle \ell_2\ i-1\rangle\\ \langle p|K_2\cdot K_3|i-1\rangle &= \langle p\ \ell_2\rangle[\ell_2\ \ell_3]\langle \ell_3\ i-1\rangle\end{aligned} \tag{4.2}$$

and $\lambda_{\ell_1} = \alpha\lambda_{i-1},\ \lambda_{\ell_4} = \beta\lambda_{i-1}$, so

$$\begin{aligned}\widehat{g}_{n:r:r';i} = &\frac{\langle i-2\ i-1\rangle\langle i-1\ i\rangle\langle i+r-1\ i+r\rangle\langle i+r+r'-1\ i+r+r'\rangle}{2\prod_{s=1}^{n}\langle s\ s+1\rangle}\frac{\langle i-1\ j\rangle^4\langle k\ i-1\rangle^4}{\langle i-1\ i\rangle\langle i-2\ i-1\rangle}\\ &\times\frac{(-K_2^2)^3}{\langle i+r-1|K_2\cdot K_3|i-1\rangle\langle i+r|K_2\cdot K_3|i-1\rangle}\\ &\times\frac{1}{\langle i+r+r'-1|K_2\cdot K_1|i-1\rangle\langle i+r+r'|K_2\cdot K_1|i-1\rangle}\\ &\times\frac{[\ell_1\ \ell_4]^3}{[\ell_4\ i-1][i-1\ \ell_1]}\left(\alpha^2\beta^2\langle i-1|\ell_3\cdot\ell_2|i-1\rangle\right).\end{aligned} \tag{4.3}$$

We will show below that the last line is simply $\left(K_{41}^2K_{12}^2 - K_1^2K_3^2\right)$. Thus we get immediately the coefficients

$$\begin{aligned}g_{n:r:r';i} = &\frac{\langle i-1\ j\rangle^4\langle k\ i-1\rangle^4\langle i+r-1\ i+r\rangle\langle i+r+r'-1\ i+r+r'\rangle(K_2^2)^3}{\prod_{s=1}^{n}\langle s\ s+1\rangle\langle i+r-1|K_2\cdot K_3|i-1\rangle}\\ &\times\frac{1}{\langle i+r|K_2\cdot K_3|i-1\rangle\langle i+r+r'-1|K_2\cdot K_1|i-1\rangle\langle i+r+r'|K_2\cdot K_1|i-1\rangle}.\end{aligned} \tag{4.4}$$

It can be shown that if we put $j=3, i-1=2, k=1$ we reproduce the all-multiplicity NMHV coefficients given in [5]. Also, our general formula matches the results of [5] for seven gluons, i.e., the coefficient c_{146} of helicity assignment $(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+)$ and c_{257} of helicity assignment $(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+)$.

Now we prove the identity involving the last line of (4.4), namely that

$$\frac{[\ell_1\ \ell_4]^3}{[\ell_4\ i-1][i-1\ \ell_1]}\left(\alpha^2\beta^2\langle i-1|\ell_3\cdot\ell_2|i-1\rangle\right) = K_{41}^2K_{12}^2 - K_1^2K_3^2. \tag{4.5}$$

To do this, we need to notice that

$$\widetilde{\lambda}_{\ell_4} = \frac{1}{\beta}\widetilde{\lambda}_{i-1} + \frac{\alpha}{\beta}\widetilde{\lambda}_{\ell_1}, \tag{4.6}$$

which can be obtained by noticing that $\ell_4 = \ell_1 + p_{i-1}$ while $p_{i-1} = \lambda_{i-1}\widetilde{\lambda}_{i-1}$, $\ell_1 = \alpha\lambda_{i-1}\widetilde{\lambda}_{\ell_1}$, $\ell_4 = \beta\lambda_{i-1}\widetilde{\lambda}_{\ell_4}$. Using this we can easily show that

$$\begin{aligned}
&\left(K_{41}^2K_{34}^2 - K_1^2K_3^2\right) = K_1^2(2p_{i-1}\cdot K_3) + K_3^2(2p_{i-1}\cdot K_1) + (2p_{i-1}\cdot K_3)(2p_{i-1}\cdot K_1)\\
&= (-)\langle \ell_1\ \ell_2\rangle[\ell_1\ \ell_2]\langle i-1\ \ell_3\rangle[i-1\ \ell_3] + (-)\langle \ell_4\ \ell_3\rangle[\ell_4\ \ell_3](-)\langle i-1\ \ell_2\rangle[i-1\ \ell_2]\\
&\quad + \langle i-1\ \ell_3\rangle[i-1\ \ell_3](-)\langle i-1\ \ell_2\rangle[i-1\ \ell_2]\\
&= -\alpha\langle i-1\ \ell_2\rangle[\ell_1\ \ell_2]\langle i-1\ \ell_3\rangle[i-1\ \ell_3] + \beta\langle i-1\ \ell_3\rangle[\ell_4\ \ell_3]\langle i-1\ \ell_2\rangle[i-1\ \ell_2]\\
&\quad - \langle i-1\ \ell_3\rangle[i-1\ \ell_3]\langle i-1\ \ell_2\rangle[i-1\ \ell_2]\\
&= -\alpha\langle i-1\ \ell_2\rangle\langle i-1\ \ell_3\rangle[\ell_1\ i-1][\ell_2\ \ell_3] = \alpha[i-1\ \ell_1]\langle i-1\ \ell_3\rangle[\ell_3\ \ell_2]\langle i-1\ \ell_2\rangle
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\frac{[\ell_1\ \ell_4]^3}{[\ell_4\ i-1][i-1\ \ell_1]}\left(\alpha^2\beta^2\langle i-1|\ell_3\cdot\ell_2|i-1\rangle\right)\\
&= \frac{\frac{1}{\beta^3}[\ell_1\ i-1]^3}{\frac{\alpha}{\beta}[\ell_1\ i-1][i-1\ \ell_1]}\alpha^2\beta^2\langle i-1\ \ell_3\rangle[\ell_3\ \ell_2]\langle \ell_2\ i-1\rangle\\
&= \alpha[i-1\ \ell_1]\langle i-1\ \ell_3\rangle[\ell_3\ \ell_2]\langle \ell_2\ i-1\rangle.
\end{aligned} \tag{4.8}$$

By similar calculations, we have found another series of coefficients of F^{3m}. where the three negative helicities $j, k, i-1$ are at the vertices $(i+r, ..., i+r+r'-1)$, $(i+r+r', ..., i-2)$ and massless leg $(i-1)$ respectively. The coefficient is given by

$$\begin{aligned}
g_{n:r:r';i} =& \frac{\langle k\ \ i-1\rangle^4\langle j|K_2\cdot K_1|i-1\rangle^4\langle i+r-1\ \ i+r\rangle\langle i+r+r'-1\ \ i+r+r'\rangle}{(-K_2^2)\prod_{s=1}^n\langle s\ s+1\rangle}\\
&\times \frac{1}{\langle i+r-1|K_2\cdot K_3|i-1\rangle\langle i+r|K_2\cdot K_3|i-1\rangle}\\
&\times \frac{1}{\langle i+r+r'-1|K_2\cdot K_1|i-1\rangle\langle i+r+r'|K_2\cdot K_1|i-1\rangle}.
\end{aligned} \tag{4.9}$$

Now we offer an example of box integrals whose quadruple cut involves fermions and scalars, in addition to gluons, circulating in the loop.

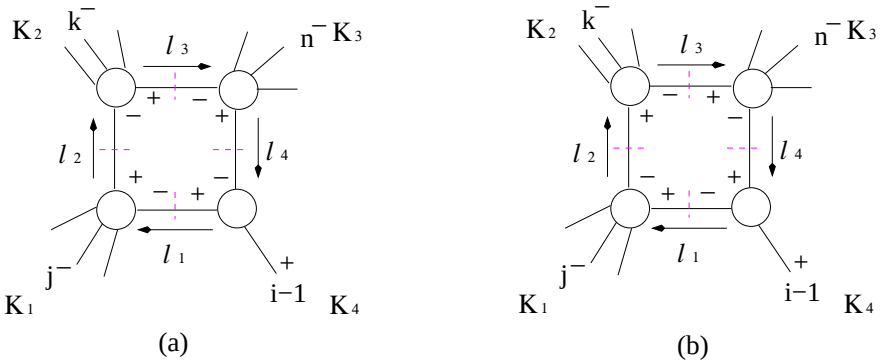

Fig. 11: Another family of three-mass scalar box integrals in an NMHV configuration.

In this case, the three negative helicities j, k, n are at vertex K_1, K_2 and K_3 respectively. There are two possible helicity configurations for loop momentum. See Figure 11. Using Ward identities, one can prove the following relations between amplitudes involving gluons and amplitudes involving fermions and scalars [24,23].

$$\begin{aligned} A(F_1^-, g_2^+, .., g_j^-, ..., F_n^+) &= \frac{\langle j\ n\rangle}{\langle j\ 1\rangle} A^{\rm MHV}(g_1^-, g_2^+, ..., g_j^-, ..., g_n^+), \\ A(S_1^-, g_2^+, .., g_j^-, ..., S_n^+) &= \frac{\langle j\ n\rangle^2}{\langle j\ 1\rangle^2} A^{\rm MHV}(g_1^-, g_2^+, ..., g_j^-, ..., g_n^+), \end{aligned} \tag{4.10}$$

we can write down the sum of configurations (a) and (b) as

$$\begin{aligned} &\frac{\langle i-2\ i-1\rangle\langle i-1\ i\rangle\langle i+r-1\ i+r\rangle\langle i+r+r'-1\ i+r+r'\rangle}{2\prod_{s=1}^{n}\langle s\ s+1\rangle} \\ &\times \frac{\langle \ell_1\ j\rangle^2\langle \ell_2\ j\rangle^2\langle \ell_2\ k\rangle^2\langle \ell_3\ k\rangle^2\langle \ell_3\ n\rangle^2\langle \ell_4\ n\rangle^2}{\langle \ell_1\ i\rangle\langle i+r-1\ \ell_2\rangle\langle \ell_2\ \ell_1\rangle\langle \ell_2\ i+r\rangle\langle i+r+r'-1\ \ell_3\rangle\langle \ell_3\ \ell_2\rangle\langle \ell_3\ i+r+r'\rangle\langle i-2\ \ell_4\rangle\langle \ell_4\ \ell_3\rangle} \\ &\times \frac{[\ell_4\ i-1]^2[\ell_1\ i-1]^2}{[i-1\ \ell_1][\ell_1\ \ell_4][\ell_4\ i-1]} \times A^{-2}\,(A-1)^4\,, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} A &= \frac{\langle j\ \ell_1\rangle\langle k\ \ell_2\rangle\langle n\ \ell_3\rangle[i-1\ \ell_1]}{\langle j\ \ell_2\rangle\langle k\ \ell_3\rangle\langle n\ \ell_4\rangle[i-1\ \ell_4]} \\ &= \frac{\langle k|K_2\cdot K_3|i-1\rangle\langle n|K_2\cdot K_1|i-1\rangle}{\langle j|K_2\cdot K_3|i-1\rangle\langle k|K_2\cdot K_1|i-1\rangle} \times \frac{\langle j|\ell_1|i-1]}{\langle n|\ell_1|i-1]}. \end{aligned} \tag{4.12}$$

After simplifying the above expression similarly to the previous examples, we obtain

$$g_{n:r:r':i} = \frac{\left(\langle k|K_2\cdot K_3|i-1\rangle\langle n|K_2\cdot K_1|i-1\rangle\langle j\ i-1\rangle - \langle j|K_2\cdot K_3|i-1\rangle\langle k|K_2\cdot K_1|i-1\rangle\langle n\ i-1\rangle\right)^4}{\langle i+r-1|K_2\cdot K_3|i-1\rangle\langle i+r|K_2\cdot K_3|i-1\rangle\langle i+r+r'-1|K_2\cdot K_1|i-1\rangle} \times \frac{\langle i+r-1\ i+r\rangle\langle i+r+r'-1\ i+r+r'\rangle}{\langle i+r+r'|K_2\cdot K_1|i-1\rangle K_2^2\langle i-1|K_2\cdot K_3|i-1\rangle^4\prod_{s=1}^n\langle s\ s+1\rangle}. \tag{4.13}$$

To simplify further, we use the following result:

$$\begin{aligned}&\langle k|K_2\cdot K_3|i-1\rangle\langle n|K_2\cdot K_1|i-1\rangle\langle j\ i-1\rangle - \langle j|K_2\cdot K_3|i-1\rangle\langle k|K_2\cdot K_1|i-1\rangle\langle n\ i-1\rangle\\ &= -\langle i-1|K_2\cdot K_3|i-1\rangle\left(\langle k|K_2\cdot K_3|n\rangle\langle j\ i-1\rangle + \langle k|K_2\cdot K_1|j\rangle\langle n\ i-1\rangle\right.\\ &\qquad\left. -\langle n\ i-1\rangle\langle j\ i-1\rangle\langle k|K_2|i-1]\right).\end{aligned} \tag{4.14}$$

We arrive at the final expression,

$$\begin{aligned}g_{n:r:r':i} = &\frac{\left(\langle k|K_2\cdot K_3|n\rangle\langle j\ i-1\rangle + \langle k|K_2\cdot (K_1+p_{i-1})|j\rangle\langle n\ i-1\rangle\right)^4}{\langle i+r-1|K_2\cdot K_3|i-1\rangle\langle i+r|K_2\cdot K_3|i-1\rangle\langle i+r+r'-1|K_2\cdot K_1|i-1\rangle}\\ &\times\frac{\langle i+r-1\ i+r\rangle\langle i+r+r'-1\ i+r+r'\rangle}{\langle i+r+r'|K_2\cdot K_1|i-1\rangle K_2^2\prod_{s=1}^n\langle s\ s+1\rangle}.\end{aligned} \tag{4.15}$$

We have checked that this expression reproduces the coefficient c_{135} of seven gluons in the helicity configuration $(1^-,2^-,3^+,4^-,5^+,6^+,7^+)$ as given in [5].

4.2. An All-Multiplicity Example of a Two-Mass-Hard Coefficient

Now we consider a series of $F^{2m\ h}$ shown in Figure 12, where both massless legs $(i-1)$ and $(i-2)$ are negative helicities.

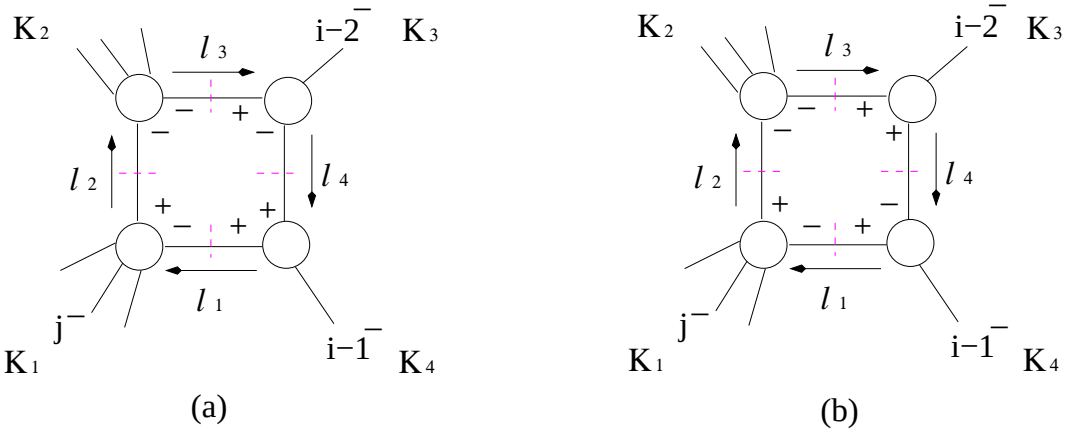

Fig. 12: Quadruple cuts of a family of two-mass-hard scalar box integrals in an NMHV configuration.

The third negative helicity j is at the vertex of $(i, ..., i+r-1)$. For this case there are two contributions. For part (a) we have

$$I_a = \frac{\langle i-3\ i-2\rangle\langle i-2\ i-1\rangle\langle i-1\ i\rangle\langle i+r-1\ i+r\rangle}{2\prod_{s=1}^{n}\langle s\ s+1\rangle} \times \frac{[\ell_1\ \ell_4]^3}{[\ell_4\ i-1][i-1\ \ell_1]}\frac{\langle \ell_1\ j\rangle^4}{\langle \ell_1\ i\rangle\langle i+r-1\ \ell_2\rangle\langle \ell_2\ \ell_1\rangle}\frac{\langle \ell_3\ \ell_2\rangle^3}{\langle \ell_2\ i+r\rangle\langle i-3\ \ell_3\rangle}\frac{\langle i-2\ \ell_4\rangle^3}{\langle \ell_3\ i-2\rangle\langle \ell_4\ \ell_3\rangle}. \tag{4.16}$$

Using similar manipulations, with $\lambda_{\ell_1} = \alpha\lambda_{i-1}$, $\lambda_{\ell_4} = \beta\lambda_{i-1}$ and the identity (4.5), which here takes the form

$$\begin{aligned}\frac{[\ell_1\ \ell_4]^3}{[\ell_4\ i-1][i-1\ \ell_1]} \times \alpha^2\beta^2\langle i-1\ \ell_3\rangle[\ell_3\ \ell_2]\langle \ell_2\ i-1\rangle &= K_{41}^2K_{12}^2 - K_1^2K_3^2 \\ &= (p_{i-1}+K_1)^2(K_1+K_2)^2\end{aligned} \tag{4.17}$$

we can read the coefficient from part (a) as

$$\frac{(K_2^2)^3\langle i-3\ i-2\rangle\langle i-2\ i-1\rangle^4\langle i-1\ j\rangle^4\langle i+r-1\ i+r\rangle/\prod_{s=1}^{n}\langle s\ s+1\rangle}{\langle i+r-1|K_2\cdot p_{i-2}|i-1\rangle\langle i+r|K_2\cdot p_{i-2}|i-1\rangle\langle i-3|K_2\cdot K_1|i-1\rangle\langle i-2|K_2\cdot K_1|i-1\rangle} \tag{4.18}$$

For part (b) we must use different spinor relations, $\lambda_{\ell_4} = \widetilde{\alpha}\lambda_{i-2}$, $\lambda_{\ell_3} = \widetilde{\beta}\lambda_{i-2}$ and an analog of the identity (4.5), namely

$$\begin{aligned}\frac{[\ell_4\ \ell_3]^3}{[\ell_3\ i-2][i-2\ \ell_4]}\widetilde{\alpha}^2\widetilde{\beta}^2\langle i-2\ \ell_2\rangle[\ell_2\ \ell_1]\langle \ell_1\ i-2\rangle &= K_{34}^2K_{41}^2 - K_4^2K_2^2 \\ &= (p_{i-2}+p_{i-1})^2(p_{i-1}+K_1)^2,\end{aligned} \tag{4.19}$$

to read out the coefficient

$$\frac{\langle j|K_1\cdot K_2|i-2\rangle^4\langle i-2\ i-1\rangle^4\langle i-1\ i\rangle\langle i+r-1\ i+r\rangle/(K_1^2\prod_{s=1}^{n}\langle s\ s+1\rangle)}{\langle i-1|K_1\cdot K_2|i-2\rangle\langle i|K_1\cdot K_2|i-2\rangle\langle i+r-1|K_1\cdot p_{i-1}|i-2\rangle\langle i+r|K_1\cdot p_{i-1}|i-2\rangle}. \tag{4.20}$$

Putting it all together, we finally have

$$\begin{aligned}d_{n:r;i} = &\\ &\frac{(K_2^2)^3\langle i-3\ i-2\rangle\langle i-2\ i-1\rangle^4\langle i-1\ j\rangle^4\langle i+r-1\ i+r\rangle/\prod_{s=1}^{n}\langle s\ s+1\rangle}{\langle i+r-1|K_2\cdot p_{i-2}|i-1\rangle\langle i+r|K_2\cdot p_{i-2}|i-1\rangle\langle i-3|K_2\cdot K_1|i-1\rangle\langle i-2|K_2\cdot K_1|i-1\rangle} \\ &+ \frac{\langle j|K_1\cdot K_2|i-2\rangle^4\langle i-2\ i-1\rangle^4\langle i-1\ i\rangle\langle i+r-1\ i+r\rangle/(K_1^2\prod_{s=1}^{n}\langle s\ s+1\rangle)}{\langle i-1|K_1\cdot K_2|i-2\rangle\langle i|K_1\cdot K_2|i-2\rangle\langle i+r-1|K_1\cdot p_{i-1}|i-2\rangle\langle i+r|K_1\cdot p_{i-1}|i-2\rangle}\end{aligned} \tag{4.21}$$

We have compared this general formula with coefficient c_{457} of seven gluons in the helicity configuration $(1^-2^-3^+4^-5^+6^+7^+)$ in [5] and found that they agree.

5. Summary

In this paper we have reduced the problem of computing the coefficient of any scalar box integral in any one-loop $\mathcal{N} = 4$ amplitude to finding solutions to the four equations

$$\ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0 \tag{5.1}$$

in $(--++)$ signature. From this set of solutions it is possible to read off the coefficient from the formula

$$\widehat{a}_\alpha = \frac{1}{|\mathcal{S}|} \sum_{\mathcal{S},J} n_J A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree}. \tag{5.2}$$

It would be interesting to perform a full classification of helicity configurations to obtain explicit formulas for all coefficients to all multiplicities.

Acknowledgments

We thank L. Dixon and E. Witten for helpful conversations. R. B. and B. F. were supported by NSF grant PHY-0070928. F. C. was supported in part by the Martin A. and Helen Chooljian Membership at the Institute for Advanced Study and by DOE grant DE-FG02-90ER40542.

Appendix A. Discontinuities of the Four-Mass Scalar Box Integral

In this section we study in detail the discontinuities of the four-mass box integral. The motivation is to allow us to perform non-trivial consistency checks on the coefficients found in the previous section and to discuss the difficulties in trying to compute them by the application of coplanar operators along the lines of [9,4].

One way to obtain the discontinuities is to compute the imaginary part of the explicit formulas for the four-mass integral in a kinematical regime chosen in order to isolate the cut of interest. Another way, which we find more intuitive, is to compute the discontinuity of the scalar box integral by cutting it directly. It turns out that the cut integral is easy to evaluate explicitly, as we now describe.

A.1. Cut Integral

Consider first the four-mass box function,

$$I^{4m} = \int d^4\ell \frac{1}{(\ell^2 + i\epsilon)((\ell - K_1)^2 + i\epsilon)((\ell - K_1 - K_2)^2 + i\epsilon)((\ell + K_4)^2 + i\epsilon)}. \tag{A.1}$$

Let us start with the discontinuity in the (K_1, K_2)-channel. The integral we have to evaluate is obtained from (A.1) by replacing the first and third propagators by the delta functions imposing the on-shell condition.

$$\Delta I^{4m}|_{K_{12}^2>0} = \int d^4\ell \frac{\delta^{(+)}(\ell^2)\delta^{(+)}((\ell - K_1 - K_2)^2)}{(\ell - K_1)^2(\ell + K_4)^2}. \tag{A.2}$$

Now we can parameterize $\ell_{a\dot{a}} = t\lambda_a\widetilde{\lambda}_{\dot{a}}$ and write the Lorentz invariant measure as follows:

$$\Delta I^{4m}|_{K_{12}^2>0} = \int_0^\infty t\ dt \int \langle\lambda\ d\lambda\rangle[\widetilde{\lambda}\ d\widetilde{\lambda}] \frac{\delta^{(+)}((\ell - K_1 - K_2)^2)}{(\ell - K_1)^2(\ell + K_4)^2}. \tag{A.3}$$

The remaining delta function can be written as

$$\delta((\ell - K_1 - K_2)^2) = \delta(t\lambda_a\widetilde{\lambda}_{\dot{a}}K_{12}^{a\dot{a}} - K_{12}^2) = \frac{1}{\lambda_a\widetilde{\lambda}_{\dot{a}}K_{12}^{a\dot{a}}}\delta(t - \frac{K_{12}^2}{\lambda_a\widetilde{\lambda}_{\dot{a}}K_{12}^{a\dot{a}}}), \tag{A.4}$$

where $K_{12} = K_1 + K_2$. Therefore the t integral can be performed to find

$$\Delta I^{4m}|_{K_{12}^2>0} = \int \langle\lambda\ d\lambda\rangle[\widetilde{\lambda}\ d\widetilde{\lambda}] \frac{K_{12}^2}{(\lambda_a\widetilde{\lambda}_{\dot{a}}K_{12}^{a\dot{a}})^2(\ell - K_1)^2(\ell + K_4)^2}. \tag{A.5}$$

The denominator can be written as

$$(\lambda_a\widetilde{\lambda}_{\dot{a}}K_{12}^{a\dot{a}})^2(\ell - K_1)^2(\ell + K_4)^2 = (Q^{a\dot{a}}\lambda_a\widetilde{\lambda}_{\dot{a}})(S^{a\dot{a}}\lambda_a\widetilde{\lambda}_{\dot{a}}), \tag{A.6}$$

with

$$\begin{aligned} Q^{a\dot{a}} &= -K_{12}^2K_1^{a\dot{a}} + K_1^2K_{12}^{a\dot{a}}, \\ S^{a\dot{a}} &= K_{12}^2K_4^{a\dot{a}} + K_4^2K_{12}^{a\dot{a}}. \end{aligned} \tag{A.7}$$

The way to evaluate this integral is to use a Feynman parametrization. This gives

$$\Delta I^{4m}|_{K_{12}^2>0} = \int_0^1 d\alpha_1 d\alpha_2 \delta(\alpha_1 + \alpha_2 - 1) \int \langle\lambda\ d\lambda\rangle[\widetilde{\lambda}\ d\widetilde{\lambda}] \frac{1}{(P^{a\dot{a}}\lambda_a\widetilde{\lambda}_{\dot{a}})^2}, \tag{A.8}$$

with

$$P^{a\dot{a}} = \alpha_1 Q^{a\dot{a}} + \alpha_2 S^{a\dot{a}}. \tag{A.9}$$

The integral over the sphere was evaluated in [14] and gives $1/P^2$. Therefore we have

$$\Delta I^{4m}|_{K_{12}^2>0} = \int_0^1 d\alpha_1 d\alpha_2 \delta(\alpha_1+\alpha_2-1)\frac{1}{\alpha_1^2 Q^2 + 2Q\cdot S\ \alpha_1\alpha_2 + \alpha_2^2 S^2}. \tag{A.10}$$

Now we can perform the integration in α_1 and α_2. The result is

$$\Delta I^{4m}|_{K_{12}^2>0} = \frac{1}{\rho K_{41}^2 K_{12}^2}\log\left(\frac{(1-\lambda_1-\lambda_2-\rho)^2}{4\lambda_1\lambda_2}\right), \tag{A.11}$$

where ρ is defined as in (1.4). We can similarly evaluate the discontinuity of the four-mass box function in the K_1-channel. This is obtained from (A.1) by replacing the first and second propagators by the delta functions imposing the on-shell condition:

$$\Delta I^{4m}|_{K_1^2>0} = \int d^4\ell \frac{\delta^{(+)}(\ell^2)\delta^{(+)}((\ell-K_1)^2)}{(\ell-K_1-K_2)^2(\ell+K_4)^2}. \tag{A.12}$$

Again, parametrize $\ell_{a\dot{a}} = t\lambda_a\widetilde{\lambda}_{\dot{a}}$ and then perform the t integral to find

$$\Delta I^{4m}|_{K_1^2>0} = \int \langle\lambda\ d\lambda\rangle[\widetilde{\lambda}\ d\widetilde{\lambda}]\frac{K_1^2}{(\lambda_a\widetilde{\lambda}_{\dot{a}}K_1^{a\dot{a}})^2(\ell-K_1-K_2)^2(\ell+K_4)^2}. \tag{A.13}$$

Here the denominator can be written as

$$(\lambda_a\widetilde{\lambda}_{\dot{a}}K_1^{a\dot{a}})^2(\ell-K_1-K_2)^2(\ell+K_4)^2 = (Q^{a\dot{a}}\lambda_a\widetilde{\lambda}_{\dot{a}})(S^{a\dot{a}}\lambda_a\widetilde{\lambda}_{\dot{a}}), \tag{A.14}$$

with

$$\begin{aligned} Q^{a\dot{a}} &= K_{12}^2 K_1^{a\dot{a}} - K_1^2 K_{12}^{a\dot{a}}, \\ S^{a\dot{a}} &= K_1^2 K_4^{a\dot{a}} + K_4^2 K_1^{a\dot{a}}. \end{aligned} \tag{A.15}$$

So we may follow exactly the same steps as in the previous case, and here we find the result

$$\Delta I^{4m}|_{K_1^2>0} = -\frac{1}{\rho K_{41}^2 K_{12}^2}\log\left(\frac{(1-\lambda_1+\lambda_2-\rho)^2}{4\lambda_2}\right), \tag{A.16}$$

where $\lambda_1, \lambda_2, \rho$ are the same expressions defined in (1.4).

A.2. Imaginary Part

As a consistency check, we can reproduce our results (A.11) and (A.16) from the imaginary part of the explicit form of the four-mass integral in the appropriate kinematical regimes.

The integral (A.1) was computed explicitly in [25] and is given in terms of the invariants $t_{ml} = -(K_m + K_{m+1} + \ldots + K_{l-1})^2$ as follows:

$$\begin{aligned} I^{4m} = \frac{1}{a(x_1 - x_2)} \sum_{j=1}^{2} (-1)^j \Bigg(& -\frac{1}{2}\ln^2(-x_j) \\ & - \mathrm{Li}_2\left(1 + \frac{t_{34} - i\epsilon}{t_{13} - i\epsilon} x_j\right) - \eta\left(-x_k, \frac{t_{34} - i\epsilon}{t_{13} - i\epsilon}\right) \ln\left(1 + \frac{t_{34} - i\epsilon}{t_{13} - i\epsilon} x_j\right) \\ & - \mathrm{Li}_2\left(1 + \frac{t_{24} - i\epsilon}{t_{12} - i\epsilon} x_j\right) - \eta\left(-x_k, \frac{t_{24} - i\epsilon}{t_{12} - i\epsilon}\right) \ln\left(1 + \frac{t_{24} - i\epsilon}{t_{12} - i\epsilon} x_j\right) \\ & + \ln(-x_j)(\ln(t_{12} - i\epsilon) + \ln(t_{13} - i\epsilon) - \ln(t_{14} - i\epsilon) - \ln(t_{23} - i\epsilon))\Bigg) . \end{aligned} \tag{A.17}$$

Here the function $\eta(x, y)$ is given by

$$\eta(x, y) = 2\pi i[\vartheta(-\mathrm{Im}\, x)\vartheta(-\mathrm{Im}\, y)\vartheta(\mathrm{Im}(xy)) - \vartheta(\mathrm{Im}\, x)\vartheta(\mathrm{Im}\, y)\vartheta(-\mathrm{Im}(xy))], \tag{A.18}$$

and x_1 and x_2 are the roots of a quadratic polynomial:

$$ax^2 + bx + c + i\epsilon d = a(x - x_1)(x - x_2), \tag{A.19}$$

with

$$\begin{aligned} a &= t_{24}t_{34}, \\ b &= t_{13}t_{24} + t_{12}t_{34} - t_{14}t_{23}, \\ c &= t_{12}t_{13}, \\ d &= t_{23}. \end{aligned} \tag{A.20}$$

The $i\epsilon$ prescription in (A.17) allows us to use this formula in any kinematical regime. The main simplification is that the proper branch of each of the functions in (A.17) is simply given by the principal branch. The formula for I^{4m} presented in [3] can be recovered from (A.17) by setting the η functions and ϵ to zero. Even though the formula looks simpler in that form, it has the disadvantage that it is not clear which branch of the various dilogarithms and logarithms compute the appropriate discontinuity.

Appendix B. Twistor Space Structure of Four-Mass Box Coefficients

In this section we study the twistor space localization of the four-mass scalar box function coefficient found in section 3.1. The way we choose to do this can also be thought of as a consistency check on the coefficient.

Consider for example the amplitude $A_{8;1}(1^-,2^-,3^-,4^-,5^+,6^+,7^+,8^+)$. The two four-mass box integrals have momenta distributed as $(12)-(34)-(56)-(78)$ and $(23)-(45)-(67)-(81)$. In the first case we find that the coefficient must be zero, since one of the tree-level amplitudes in the quadruple cut diagram is necessarily zero. This is essentially equivalent to the observation of [5], where the same conclusion was derived from a triple cut.

In order to study the second four-mass integral, consider the cut in the (2345)-channel. This cut is given by

$$C_{2345} = \int d\mu \; A_6^{\rm tree}(\ell_1, 2^-, 3^-, 4^-, 5^+, \ell_2) A_6^{\rm tree}(-\ell_2, 6^+, 7^+, 8^+, 1^-, -\ell_1). \tag{B.1}$$

There are four contributions to this cut depending on the helicities (h_{ℓ_1}, h_{ℓ_2}) of the particles running in the cut propagators. If $(h_{\ell_1}, h_{\ell_2}) = (-,-)$ then we get zero. If $(h_{\ell_1}, h_{\ell_2}) = (-,+)$ or $(h_{\ell_1}, h_{\ell_2}) = (+,-)$, then the whole $\mathcal{N}=4$ supermultiplet contributes. In this case, both tree-level amplitudes are very simple; they become a mostly-minus MHV and a mostly-plus MHV, respectively. These contributions can be easily written as

$$\frac{\langle 1|(2+3+4)|5]^4}{((p_1+p_2+p_3+p_4)^2)^4} \, [C_{1234}]_{i\to i+1} \, . \tag{B.2}$$

Since the four-mass integral does not contribute to the cut C_{1234}, then it does not contribute to this part of the cut C_{2345} either.

Finally, we have the case with $(h_{\ell_1}, h_{\ell_2}) = (+,+)$. In this case, only gluons can propagate; the complication arises from the fact that both tree-level amplitudes are next-to MHV six-gluon amplitudes.

It turns out that we cannot use collinear operators to extract information from this cut. The reason is that a collinear operator does not localize the integral. We need a coplanar operator.

A coplanar operator is defined as follows [11].

$$K_{ijkl} = \langle i\; j\rangle[\widetilde{\partial}_k\; \widetilde{\partial}_l] + \langle j\; k\rangle[\widetilde{\partial}_i\; \widetilde{\partial}_l] + \langle k\; i\rangle[\widetilde{\partial}_j\; \widetilde{\partial}_l] + \langle k\; l\rangle[\widetilde{\partial}_i\; \widetilde{\partial}_j] + \langle i\; l\rangle[\widetilde{\partial}_j\; \widetilde{\partial}_k] + \langle j\; l\rangle[\widetilde{\partial}_k\; \widetilde{\partial}_i], \tag{B.3}$$

where

$$(\widetilde{\partial}_i)_{\dot\alpha} = \frac{\partial}{\partial \widetilde{\lambda}_i^{\dot\alpha}}. \tag{B.4}$$

In this case we could use K_{2345} or K_{6781} to produce rational functions.

Recall that the cut can also be written as the discontinuity of the amplitude in the $(2345)-$channel.

$$C_{2345} = \Delta A_8^{1-\text{loop}} = \ldots + \widehat{f}\Delta I^{4m}. \tag{B.5}$$

We have only explicitly written the term from the four-mass integral. Once we apply either of the two coplanar operators K on the cut integral, we produce a rational function. On the other hand, acting with K on (B.5) produces

$$KC_{2345} = \ldots + K(\widehat{f}\Delta I^{4m}). \tag{B.6}$$

Following the arguments of [9], one can show that the terms in the ellipses, which come from $1m$, $2m$ and $3m$ scalar box integrals, are rational functions. Therefore we conclude that

$$K(\widehat{f}\Delta I^{4m}) \tag{B.7}$$

must be rational.

But we can go even farther. In the previous section, we found that

$$\Delta I^{4m} = \frac{1}{\rho K_{41}^2 K_{12}^2} \log\left(\frac{(1-\lambda_1-\lambda_2-\rho)^2}{4\lambda_1\lambda_2}\right). \tag{B.8}$$

Therefore, when none of the derivatives in K act on the logarithm, we find a term of the form

$$K\left(\frac{\widehat{f}}{\rho K_{41}^2 K_{12}^2}\right) \times \log\left(\frac{(1-\lambda_1-\lambda_2-\rho)^2}{4\lambda_1\lambda_2}\right). \tag{B.9}$$

The only way this is consistent with (B.7) being rational is that

$$K\left(\frac{\widehat{f}}{\rho K_{41}^2 K_{12}^2}\right) = 0. \tag{B.10}$$

Recall that $f = -2\widehat{f}/(\rho K_{41}^2 K_{12}^2)$ is the definition of the four-mass box function coefficient, which we claimed would have a simple twistor structure configuration. Indeed, (B.10) is the reason for our claim.

So we have found that $K_{2345} f_2 = K_{6781} f_2 = 0$. Note that this four-mass box integral also has a cut in the $(4567)-$channel. Using the same logic we find that $K_{4567} f_2 = K_{8123} f_2 = 0$.

The conclusion is then that all gluons in two adjacent corners of the four-mass box function coefficient are localized on a plane. The most general configuration consistent with this picture is shown in Figure 13.

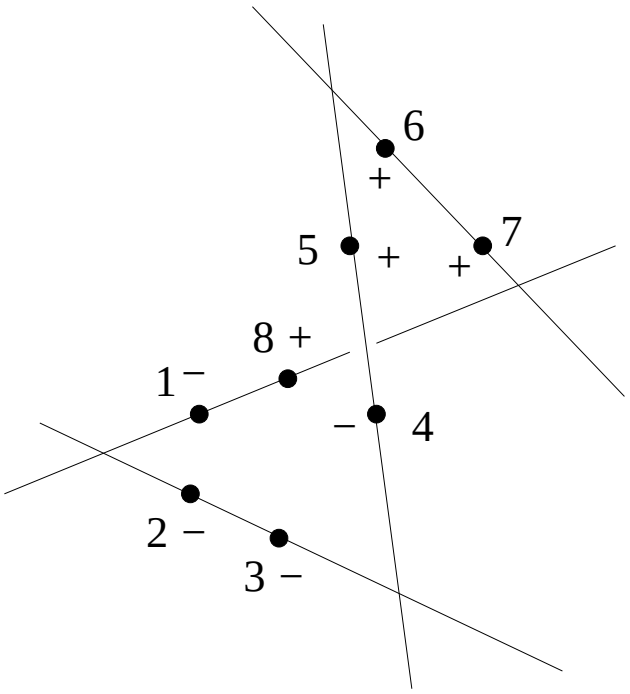

Fig. 13: The twistor space structure of the four-mass coefficient f_2. The four lines are not all coplanar.

The consistency checks we have run on the coefficient found in section 3.1 are the following: f_2 is annihilated by the coplanar operators mentioned before, and (B.7) is a rational function.

At this point we must comment on why studying this cut does not give an efficient way of computing f_2. After all, computing the rational function $K_{2345}C_{2345}$ from the cut integral representation of C_{2345} is very simple. Moreover, it turns out that all other coefficients contributing to this cut can be computed from cuts in three-particle channels, along with linear equations from the infrared singularities, using the method of [4]. Therefore, all reduces to

$$W = K(\widehat{f}\Delta I^{4m}), \tag{B.11}$$

where W is a known rational function, and ΔI^{4m} is also known.

Let us write K schematically as $\mathcal{O}_1\mathcal{O}_2$. Then, expanding the right hand side of (B.11) we find

$$W = K(\widehat{f})\Delta I^{4m} + \mathcal{O}_1(\widehat{f})\mathcal{O}_2(\Delta I^{4m}) + \mathcal{O}_2(\widehat{f})\mathcal{O}_1(\Delta I^{4m}) + \widehat{f}K(\Delta I^{4m}). \tag{B.12}$$

The problem arises when one notices that the only term that has no derivatives acting on $\widehat{f}$ vanishes. Indeed, it not difficult to show with the help of a symbolic manipulation program that $K(\Delta I^{4m}) = 0$.

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